Mixed Convection in a Composite System Bounded by Vertical Walls

N. Srivastava¹ and A. K. Singh²

Department of Mathematics, Banaras Hindu University, Varanasi-221005, India

Email: ¹neetusri_81@rediffmail.com, ²ashok@bhu.ac.in

(Received March 30, 2009; accepted September 6, 2009)

Abstract

A combined convection process between two parallel vertical infinite walls, containing an incompressible viscous fluid layer and a fluid saturated porous layer has been presented analytically. There is a vertical axial variation of temperature in the upward direction along the walls. The Brinkman extended Darcy model is applied to describe the momentum transfer in the porous region. The viscosity of the fluid layer and the effective viscosity of the porous layer are assumed to be different. Also the thermal conductivities of both fluid and porous layers are assumed to be different. The graphs and tables have been used to distinguish the influence of distinct parameters on the velocity and skin-friction. It is determined that the velocity is intensified on making greater the temperature difference between the walls while increment in the viscosity ratio (porous/fluid) parameter diminishes the velocity of the fluid. It has been observed that the numerical values of the skin-frictions have an increasing tendency with the increment in the values of temperature difference between the walls while decreasing tendency with the increment in the viscosity ratio parameter (porous/fluid).

Keywords: Mixed convection, composite system, effective viscosity, porous media, Brinkman model

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>temperature gradient along the wall</td>
</tr>
<tr>
<td>Da</td>
<td>Darcy number</td>
</tr>
<tr>
<td>d'</td>
<td>distance of interface</td>
</tr>
<tr>
<td>d</td>
<td>distance of interface in non-dimensional form</td>
</tr>
<tr>
<td>g</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>H</td>
<td>distance between the vertical walls</td>
</tr>
<tr>
<td>k'</td>
<td>permeability of the porous medium</td>
</tr>
<tr>
<td>P'</td>
<td>pressure</td>
</tr>
<tr>
<td>Q</td>
<td>constant including pressure gradient term</td>
</tr>
<tr>
<td>Ra</td>
<td>Rayleigh number</td>
</tr>
<tr>
<td>Rc</td>
<td>ratio of thermal conductivities</td>
</tr>
<tr>
<td>Ru</td>
<td>ratio of effective viscosity to the dynamic viscosity</td>
</tr>
<tr>
<td>T₀'</td>
<td>reference temperature</td>
</tr>
<tr>
<td>T₁'</td>
<td>temperature at the wall y' = 0</td>
</tr>
<tr>
<td>u'</td>
<td>velocity along the x'-direction</td>
</tr>
<tr>
<td>u</td>
<td>velocity along x-direction in non-dimensional form</td>
</tr>
<tr>
<td>x'</td>
<td>vertical coordinate</td>
</tr>
<tr>
<td>x</td>
<td>vertical coordinate in non-dimensional form</td>
</tr>
<tr>
<td>y'</td>
<td>horizontal coordinate</td>
</tr>
<tr>
<td>y</td>
<td>horizontal coordinate in non-dimensional form</td>
</tr>
<tr>
<td>α</td>
<td>thermal diffusivity</td>
</tr>
<tr>
<td>β</td>
<td>coefficient of thermal expansion</td>
</tr>
<tr>
<td>μf</td>
<td>dynamic viscosity of the fluid</td>
</tr>
<tr>
<td>μ eff</td>
<td>effective viscosity of the porous region</td>
</tr>
<tr>
<td>θ</td>
<td>temperature in non-dimensional form</td>
</tr>
<tr>
<td>ρ</td>
<td>density</td>
</tr>
</tbody>
</table>

Greek symbols

- α: thermal diffusivity
- β: coefficient of thermal expansion
- μf: dynamic viscosity of the fluid
- μ eff: effective viscosity of the porous region
- θ: temperature in non-dimensional form
- ρ: density

Subscripts

- f: fluid layer
- p: porous layer
1. INTRODUCTION

The phenomena of mixed or combined convection arise when both free and forced convection simultaneously occur. Free convection means the motion which arises due to buoyancy effects while forced convection results due to any external force. An analytical solution of mixed convective flow between vertical parallel walls for higher Rayleigh number has been presented by Beckett and Friend (1984). A study of mixed convection flows between parallel plate channels has been presented by Aung and Worku (1986). An exact analytical solution of mixed convective flow on a permeable vertical cylinder in a saturated porous medium has been given by Ramanahiah and Malarvizhi (1990). Singh et al. (1993) have presented numerically the three dimensional free convection in a cavity as a result of side heating. An analytical study of natural convective flow in a composite system containing fluid and porous layers between two vertical walls has been presented by Paul et al. (1998). Paul et al. (1999) have further extended in case of unsteady natural convective flow.

The problem of mixed convection in a porous medium bounded by two vertical walls has been done by Mishra et al. (2002). Nobari and Beshkani (2007) have studied numerically the mixed convective flow in a vertical channel by applying finite difference method based on projection algorithm. The mixed convective flow between vertical parallel plates has been solved numerically by Guillet et al. (2007). A numerical study of mixed convective flow in a porous medium has been given by Zanchini (2008) has given an analytical solution of mixed convection in a composite system containing fluid and porous layers when the temperature of the walls varies vertically upward. Brinkman extended Darcy model is used to model the flow in porous region. For fully developed laminar flow, the velocity has only one component in vertical direction. The viscosity of the fluid layer and the effective viscosity of the porous layer are taken to be different. Three different analytical solutions of the model have been obtained depending on the values of Darcy number, viscosity ratio parameter, thermal conductivity ratio parameter and Rayleigh number. Finally effects of various physical parameters have been shown by using graphs and tables.

2. MATHEMATICAL FORMULATION

In the given problem a steady fully developed laminar free convective flow between two infinite vertical walls filled with a fluid layer and a fluid saturated porous layer is considered as shown in Fig. 1. The interface of fluid and porous layers is taken permeable so that fluid can flow from one layer to other. The $x'$-axis is taken in the vertical direction while $y'$-axis is taken in the horizontal direction. The walls at $y' = 0$ and $y' = H$ are maintained at the temperatures $T_0 + Ax'$ and $T_1 + Ax'$ respectively, where $x'$ is the distance measured vertically in the upward direction. Under usual Boussinesq’s approximation, the governing equations in the reference of the considered problem in non-dimensional form are derived as follows:

For fluid region (Beckett and Friend (1984)):

$$\frac{d^2 u_f}{dy'^2} + Ra \theta_f = -1, \quad (1)$$

$$\frac{d \theta_f}{dy'} - u_f = 0, \quad (2)$$

For porous region (Mishra et al. (2002)):

$$Re \frac{d^2 u_p}{dy'^2} + Ra \theta_p - \frac{1}{Da} u_p = -1, \quad (3)$$
The corresponding boundary and matching conditions in non-dimensional form are acquired as follows (Singh et al. (1993)):

1. When eliminating \(\theta_p\) from Eqs. (1) and (2), we get a fourth order differential equation in \(u_f\) as

\[
Ra u_f'' + Ra u_f' = 0,
\]

while eliminating \(\theta_p\) from Eqs. (3) and (4) a fourth order differential equation in \(u_p\) is obtained as

\[
du_p'' + \frac{1}{Da} du_p' + \frac{Ra}{Re} u_p' = 0,
\]

whose auxiliary roots are obtained as given below:

\[
m_1, m_2 = \pm \sqrt{\frac{1+S}{2 Da Ra}},
\]

\[
m_3, m_4 = \pm \sqrt{\frac{1-S}{2 Da Ra}},
\]

where

\[
S = \sqrt{1 - \frac{4 Ra Re Da^2}{Re}}.
\]

It is obvious from the auxiliary roots described in Eq. (9), that the solution for the velocity and temperature fields depends on the values of \(Ra\), \(RV\), \(Da\) and \(Re\) and there arises three different cases which are as follows:

Case1. When \(1 - \frac{4 Ra Re Da^2}{Re} > 0\);

In this case \(S\) will be a real number. Solving Eqs. (7) and (8) with their proper boundary conditions the solution for \(u_f, \theta_f, u_p\) and, \(\theta_p\) are obtained as given below:

\[
u_f = e^{Ny}(C_1 \cos Ny + C_2 \sin Ny) + e^{-Ny}(C_5 \cos Ny + C_4 \sin Ny),
\]

\[
\theta_f = -\frac{1}{Ra} - \frac{2N^2}{Ra}[e^{Ny}(C_2 \cos Ny - C_1 \sin Ny) + e^{-Ny}(C_3 \sin Ny - C_4 \cos Ny)],
\]

\[
u_p = (C_5 \cos h_1y + C_6 \sin h_1y) + (C_7 \cos h_2y + C_8 \sin h_2y),
\]

\[
\theta_p = -\frac{1}{Ra} - \frac{RV}{Ra}[C_3 h_1^2 \cos h_1y + C_6 h_2^2 \sin h_1y]
\]

\[
+ C_5 \sin (h_1y) + C_6 \sin (h_1y) + C_7 \cos (h_2y) + C_8 \sin (h_2y)].
\]

The above solutions are valid only for \(0 < S < 1\).

Case2. When \(1 - \frac{4 Ra Re Da^2}{Re} = 0\);

In this case, the velocity and temperature fields in the fluid and porous region with the suitable boundary conditions are derived as

\[
u_f = e^{Ny}(C_1 \cos Ny + C_2 \sin Ny) + e^{-Ny}(C_5 \cos Ny + C_4 \sin Ny),
\]

\[
\theta_f = -\frac{1}{Ra} - \frac{2N^2}{Ra}[e^{Ny}(C_2 \cos Ny - C_1 \sin Ny) + e^{-Ny}(C_3 \sin Ny - C_4 \cos Ny)],
\]

\[
u_p = (C_5 + C_6) \sinh my + (C_7 + C_8) \cosh my,
\]

\[
\theta_p = -\frac{1}{Ra} - \frac{RV}{Ra}[(m^2(C_1 + C_2y) + 2mC_4)
\]

\[
\cos hy + (m^2(C_1 + C_2y) + 2mC_4) \sinh my]
\]

\[
+ \frac{1}{Da Ra}[(C_1 + C_2y) \cosh my + (C_1 + C_4y) \sinh my].
\]

Case3. When \(1 - \frac{4 Ra Re Da^2}{Re} < 0\);

S will be imaginary in this case. The expressions for the velocity and temperature fields in the fluid and porous region with the corresponding boundary conditions are

\[
u_f = e^{Ny}(C_1 \cos Ny + C_2 \sin Ny) + e^{-Ny}(C_5 \cos Ny + C_4 \sin Ny),
\]

\[
\theta_f = -\frac{1}{Ra} - \frac{2N^2}{Ra}[e^{Ny}(C_2 \cos Ny - C_1 \sin Ny) + e^{-Ny}(C_3 \sin Ny - C_4 \cos Ny)],
\]

\[
\theta_f = -\frac{1}{Ra} - \frac{2N^2}{Ra}[e^{Ny}(C_2 \cos Ny - C_1 \sin Ny) + e^{-Ny}(C_3 \sin Ny - C_4 \cos Ny)].
\]
\[ u_y = e^{ay}(C_5 \cos \beta y + C_6 \sin \beta y) + e^{-ay}(C_7 \cos \beta y + C_8 \sin \beta y), \quad (20) \]

\[ \theta_y = -\frac{1}{Ra} + \frac{1}{Da} \left( e^{ay} \beta(C_5 \cos \beta y + C_6 \sin \beta y) + e^{-ay}(C_7 \cos \beta y + C_8 \sin \beta y) \right) \]

By using the Eqs. (10), (12), (14), (16), (18) and (20), expressions for the skin frictions in all three cases are derived as follows:

**Case 1:** When \( 1 - \frac{4RaRaDa}{Re} > 0; \)

\[ \tau_1 = \left( \frac{du_y}{dy} \right)_{y=0} = N(C_1 + C_2 + C_4 - C_3), \quad (22) \]

\[ \tau_2 = -\left( \frac{du_y}{dy} \right)_{y=1} = -(C_3 h_1 \sin h_1 + C_6 h_1 \cosh h_2) + (C_3 h_2 \sin h_2 + C_6 h_2 \cosh h_2). \quad (23) \]

**Case 2:** When \( 1 - \frac{4RaRaDa}{Re} = 0; \)

\[ \tau_1 = \left( \frac{du_y}{dy} \right)_{y=0} = -N(C_1 + C_2 + C_4 - C_3), \quad (24) \]

\[ \tau_2 = \left( \frac{du_y}{dy} \right)_{y=1} = (C_3 + C_4) m + C_6 \sin m + (C_3 + C_6) m + C_7 \cosh m. \quad (25) \]

**Case 3:** When \( 1 - \frac{4RaRaDa}{Re} < 0; \)

\[ \tau_1 = \left( \frac{du_y}{dy} \right)_{y=0} = N(C_1 + C_2 + C_4 - C_3), \quad (26) \]

\[ \tau_2 = -\left( \frac{du_y}{dy} \right)_{y=1} = e^{ay}(C_2 \alpha \cos \beta - \beta \sin \beta) + C_8(\alpha \sin \beta + \beta \cos \beta - \sin \beta - C\beta \sin \beta + 2\alpha \beta \cos \beta) \quad (27) \]

The parameters used in the above equations are mentioned in the appendix.

### 3. RESULTS AND DISCUSSION

The influence of different physical parameters on the velocity field is depicted in the Figs. 2-3 for the case \( S > 0 \) while in the Figs. 4-6 for \( S = 0 \) and finally for the case \( S < 0 \) in the Figs. 7-11. In Fig. 2 the effect of thermal conductivity ratio parameter \( R_c \) and viscosity ratio parameter \( R_v \) on the velocity field is shown. From the figure it can be clearly observed that the velocity is showing decreasing behavior with the increment in the values of \( R_v \) for all the values of Darcy number and this is expected due to impact of more viscous force. For \( Da = 10^{-2} \), we can see that the effect of viscosity ratio parameter is negligible in the porous region. There is a decrement in the velocity on increasing the values of \( R_c \) in the fluid region while there is an increment in the velocity on increasing the values of \( R_c \) in the porous region. This phenomenon may occur due to more diffusion of heat in porous layer than fluid layer. In Fig. 3, the impact of pressure gradient constant \( Q \), on the velocity field is held to view. By examining the figure it has been noticed that the velocity becomes greater when the values of \( Q \) grow i.e., when the temperature difference between two walls is increasing.

This is attributed to fact that velocity increases due to increase in the buoyancy force. In Fig. 4, the effect on the velocity field for the different values of \( R_c \), \( R_v \) and \( Ra \) has been displayed when \( S = 0 \). The figure manifestly notifies that on increasing the value of \( R_v \), the velocity decreases for all the values of Darcy number.

In Fig. 5, the relation between the fluid layer width \( d \) and the velocity has been presented to view. The figure completely notifies that with the increment in the fluid layer width \( d \) the velocity has growing nature for all the values of Darcy number. But for \( Da = 0.5 \), in the porous region there is a negligible influence of increment in fluid layer width. In Fig. 6, the variation in the velocity due to variation in the values of \( Q \) has been shown. It is clear from the figure that when \( Q = 0 \), that is when the temperature difference between two walls is zero then the flow is in the upward direction while for \( Q > 0 \), the flow is in the reverse direction.

In Fig. 7, the velocity profiles are shown for different values of \( R_c \) and \( R_v \). It is remarkable from the figure that the effect of \( R_v \) on the velocity is to decrease it for all considered values of Darcy number. In this case also the velocity has a decreasing tendency in the fluid region while it is showing an increasing nature in the porous region on increasing the values of \( R_c \). Also velocity is more apparent for \( Da = 10^{-1} \) than that of \( Da = 10^{-2} \). In Fig. 8, the effect of Rayleigh number \( Ra \) on the velocity field has been exhibited. It is quite remarkable from the figure that there is a decrement in the velocity due to increment in the values of \( Ra \). In Fig. 8, the relation of velocity with the fluid layer width has been hold to view. From the figure it can be easily seen that the velocity varies in an increasing manner on increasing the fluid layer width. Figures 10 and 11 are given to describe the effect of \( Q \) on the velocity when \( Q \) hold the values close to zero and when \( Q \) hold the values greater than zero respectively. From both the Figs. 9 and 10, it is obvious that the velocity has increasing tendency with the increment in the values of \( Q \) for all the values of Darcy number.

At last, the numerical values of the skin-friction are computed on both the walls. Here, \( \tau_1 \) and \( \tau_2 \) represent...
the numerical values of skin-friction on the walls \( y = 0 \) and \( y = 1 \) respectively and they are given in Tables 1, 2 and 3 for the different cases \( S > 0, S = 0, \) and \( S < 0 \) respectively. From the Table 1, it is observed that on increasing the values of \( Rv \) and \( Ra \) both \( \tau_1 \) and \( \tau_2 \) are decreasing while increasing due to increment in the values of Darcy number, fluid layer width \( d \) and pressure gradient constant \( Q. \) The increment in the values of \( Rc \) makes decrement in \( \tau_1 \) while it makes increment in \( \tau_2 \) for all the values of Darcy number. From Table 2, it can be clearly viewed that the values of both \( \tau_1 \) and \( \tau_2 \) increase with the fluid layer width \( d \) and pressure gradient constant \( Q. \) From Table 3, it can be notified that both \( \tau_1 \) and \( \tau_2 \) have increasing tendency when the Darcy number \( Da, \) fluid layer width \( d \) and pressure gradient constant \( Q, \) tend to increment. With the increment in the values of \( Rc, \) \( \tau_1 \) decreases while \( \tau_2 \) increases. In this case, both \( \tau_1 \) and \( \tau_2 \) are showing decreasing behavior on increasing the values of \( Rv \) and \( Ra. \)

4. CONCLUSION

An analytical solution of mixed convection in a composite system containing a fluid layer and a porous layer bounded by vertical walls has been attained. It is resolved from the discussion that the velocity in both fluid and porous regions has decreasing tendency with the viscosity ratio parameter for all the cases which is due to fact that velocity decreases due to more viscous forces. The effect of the thermal conductivity ratio parameter is to enhance the velocity in the porous region while to diminish the velocity in the fluid region for all the values of Darcy number. Lastly, it has been concluded that numerical values of the skin friction exhibit an increasing behavior with the increment in the values of temperature difference between the walls, Darcy number and fluid layer width while the numerical values of skin friction are decreasing due to the viscosity ratio parameter and Rayleigh number.

REFERENCES


Fig. 4. Velocity profiles for different values of Re, Rv and Ra (Case 2)

Fig. 5. Velocity profiles for different values of d (Case 2)

Fig. 6. Velocity profiles for different values of Q (Case 2)

Fig. 7. Velocity profiles for different values of Re and Rv (Case 3)

Fig. 8. Velocity profiles for different values of Ra (Case 3)

Fig. 9. Velocity profiles for different values of d (Case 3)

Fig. 10. Velocity profiles for different values of Q (Case 3)

Fig. 11. Velocity profiles for different values of Q (Case 3)
### Numerical values of skin frictions

**Table 1** Case 1

<table>
<thead>
<tr>
<th>Da</th>
<th>Q</th>
<th>Ra</th>
<th>Rv</th>
<th>Rc</th>
<th>d</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>0.0</td>
<td>10</td>
<td>1.3</td>
<td>1.5</td>
<td>0.5</td>
<td>0.386669</td>
<td>0.254900</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.8</td>
<td>1.5</td>
<td>0.5</td>
<td>0.386578</td>
<td>0.220868</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.401966</td>
<td>0.260029</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.380595</td>
<td>0.260339</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>15</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.366278</td>
<td>0.236119</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>22</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.362220</td>
<td>0.225121</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.348182</td>
<td>0.215806</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.7</td>
<td>0.422484</td>
<td>0.280347</td>
</tr>
<tr>
<td>1.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.998385</td>
<td>1.153310</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.998580</td>
<td>0.991616</td>
<td></td>
</tr>
<tr>
<td>(10^{-2})</td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.296468</td>
<td>0.085167</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.296415</td>
<td>0.086007</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>15</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.294816</td>
<td>0.085956</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>22</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.290567</td>
<td>0.084347</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.206724</td>
<td>0.081821</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.7</td>
<td>0.393193</td>
<td>0.116738</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.665799</td>
<td>0.464376</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>10</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>1.034200</td>
<td>0.838864</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2** Case 2

<table>
<thead>
<tr>
<th>Da</th>
<th>Q</th>
<th>Ra</th>
<th>Rv</th>
<th>Rc</th>
<th>d</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>0.0</td>
<td>25</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.206240</td>
<td>0.870637</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>25</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.392751</td>
<td>1.056768</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>25</td>
<td>1.5</td>
<td>1.5</td>
<td>0.7</td>
<td>0.964422</td>
<td>1.627459</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>25</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.512209</td>
<td>1.323730</td>
</tr>
<tr>
<td>1.0</td>
<td>25</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.631668</td>
<td>1.590671</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>16.6</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.166228</td>
<td>0.950956</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>33.3</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>2.100763</td>
<td>2.803604</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>18.7</td>
<td>2.0</td>
<td>1.5</td>
<td>0.5</td>
<td>2.929225</td>
<td>1.237157</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>20.8</td>
<td>1.5</td>
<td>1.8</td>
<td>0.5</td>
<td>0.886599</td>
<td>1.533033</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.560225</td>
<td>0.532392</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.727771</td>
<td>0.551913</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.7</td>
<td>0.974949</td>
<td>0.578088</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.929358</td>
<td>0.687462</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>1.130945</td>
<td>0.823012</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>133.3</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.537431</td>
<td>0.488496</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>20.8</td>
<td>1.5</td>
<td>1.8</td>
<td>0.5</td>
<td>0.563640</td>
<td>0.497770</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3** Case 3

<table>
<thead>
<tr>
<th>Da</th>
<th>Q</th>
<th>Ra</th>
<th>Rv</th>
<th>Rc</th>
<th>d</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>0.0</td>
<td>100</td>
<td>1.0</td>
<td>1.5</td>
<td>0.5</td>
<td>0.383112</td>
<td>0.386109</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>2.0</td>
<td>1.5</td>
<td>0.5</td>
<td>0.379016</td>
<td>0.283202</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.471167</td>
<td>0.236866</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.372853</td>
<td>0.355647</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>500</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.309017</td>
<td>0.226889</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>900</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.291173</td>
<td>0.198156</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.326804</td>
<td>0.249637</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.7</td>
<td>0.587187</td>
<td>0.941567</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>4.314726</td>
<td>8.213052</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>100</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>8.246530</td>
<td>16.104570</td>
<td></td>
</tr>
<tr>
<td>(10^{-2})</td>
<td>0.0</td>
<td>10000</td>
<td>1.0</td>
<td>1.5</td>
<td>0.5</td>
<td>0.061075</td>
<td>0.078267</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10000</td>
<td>2.0</td>
<td>1.5</td>
<td>0.5</td>
<td>0.063788</td>
<td>0.061201</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10000</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.062881</td>
<td>0.054239</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10000</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.061432</td>
<td>0.076134</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>12000</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.059315</td>
<td>0.056257</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>14000</td>
<td>1.5</td>
<td>1.5</td>
<td>0.5</td>
<td>0.060947</td>
<td>0.047887</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>10000</td>
<td>1.5</td>
<td>1.5</td>
<td>0.3</td>
<td>0.139523</td>
<td>0.047492</td>
</tr>
</tbody>
</table>
Case 1:

\[ S = \sqrt{1 - \frac{4 \alpha Rv Du^2}{Rc}}, \quad A = \frac{1 + S}{2Rv}, \]

\[ N = \frac{\sqrt{h_1}}{2}, \quad B = \frac{1 - S}{2Rv}, \]

\[ h_1 = \sqrt{A}, \quad h_2 = \sqrt{B}, \]

\[ h_3 = h_1 d, \quad h_4 = h_2 d, \]

\[ h_5 = N d, \quad h_7 = 2N^2, \]

\[ r_1 = \cosh(h_1), \quad r_2 = \sinh(h_1), \]

\[ r_3 = \sinh(h_2), \quad r_4 = \sinh(h_2), \]

\[ r_5 = e^{h_6} \cos(h_6), \quad r_6 = e^{h_6} \sin(h_6), \]

\[ r_7 = e^{-h_6} \cos(h_6), \quad r_8 = e^{-h_6} \sin(h_6), \]

\[ r_{12} = \sinh(h_4), \]

\[ r_{34} = Ne^{h_6} \{ \cos(h_6) - \sin(h_6) \}, \]

\[ r_{35} = Ne^{h_6} \{ \cos(h_6) + \sin(h_6) \}, \]

\[ r_{36} = -Ne^{-h_6} \{ \cos(h_6) + \sin(h_6) \}, \]

\[ r_{37} = Ne^{-h_6} \{ \cos(h_6) - \sin(h_6) \}, \]

\[ r_{38} = Rvh_1 \sinh(h_3), \]

\[ r_{39} = Rvh_1 \cosh(h_3), \]

\[ r_{40} = -\frac{1}{Rv} \cosh(h_2), \]

\[ r_{41} = \frac{1}{Rv} \sinh(h_2), \]

\[ r_{42} = \frac{1}{Rv} \cosh(h_2), \]

\[ r_{43} = \frac{1}{Rv} \sinh(h_2), \]

\[ r_{44} = \frac{1}{Rv} \cosh(h_2), \]

\[ r_{45} = \frac{1}{Rv} \sinh(h_2), \]

\[ r_{46} = \frac{1}{Rv} \cosh(h_2), \]

\[ r_{47} = \cosh(h_2), \]

\[ r_{48} = \sinh(h_2), \]

\[ r_{49} = \sinh(h_2), \]

\[ r_{50} = \cosh(h_2), \]

\[ r_{51} = \sinh(h_2), \]

\[ r_{52} = \sinh(h_2), \]

\[ r_{53} = \cosh(h_2), \]

\[ r_{54} = \cosh(h_2), \]

\[ r_{55} = \cosh(h_2), \]

\[ r_{56} = \sinh(h_2), \]

\[ r_{57} = \sinh(h_2), \]

\[ r_{58} = \sinh(h_2), \]

\[ r_{59} = \cosh(h_2), \]

\[ r_{60} = \cosh(h_2), \]

\[ r_{61} = \cosh(h_2), \]

\[ r_{62} = \sinh(h_2), \]

\[ r_{63} = \sinh(h_2), \]

\[ r_{64} = \sinh(h_2), \]

\[ r_{65} = \cosh(h_2), \]

\[ r_{66} = \cosh(h_2), \]

\[ r_{67} = \cosh(h_2), \]

\[ r_{68} = \sinh(h_2), \]

\[ r_{69} = \sinh(h_2), \]

\[ r_{70} = \sinh(h_2), \]

\[ r_{71} = \cosh(h_2), \]

\[ r_{72} = \cosh(h_2), \]

\[ r_{73} = \cosh(h_2), \]

\[ r_{74} = \cosh(h_2), \]

\[ r_{75} = \cosh(h_2), \]

\[ r_{76} = \cosh(h_2), \]

\[ r_{77} = \cosh(h_2), \]

\[ r_{78} = \cosh(h_2), \]

\[ r_{79} = \cosh(h_2), \]

\[ r_{80} = \cosh(h_2), \]

\[ r_{81} = \cosh(h_2), \]

\[ r_{82} = \cosh(h_2), \]

\[ r_{83} = \cosh(h_2), \]

\[ r_{84} = \cosh(h_2), \]

\[ r_{85} = \cosh(h_2), \]

\[ r_{86} = \cosh(h_2), \]

\[ r_{87} = \cosh(h_2), \]

\[ r_{88} = \cosh(h_2), \]

\[ r_{89} = \cosh(h_2), \]

\[ r_{90} = \cosh(h_2), \]

\[ r_{91} = \cosh(h_2), \]

\[ r_{92} = \cosh(h_2), \]

\[ r_{93} = \cosh(h_2), \]

\[ r_{94} = \cosh(h_2), \]

\[ r_{95} = \cosh(h_2), \]

\[ r_{96} = \cosh(h_2), \]

\[ r_{97} = \cosh(h_2), \]

\[ r_{98} = \cosh(h_2), \]

\[ r_{99} = \cosh(h_2), \]

\[ r_{100} = \cosh(h_2), \]

\[ r_{101} = \cosh(h_2), \]

\[ r_{102} = \cosh(h_2), \]

\[ r_{103} = \cosh(h_2), \]

\[ r_{104} = \cosh(h_2), \]

\[ r_{105} = \cosh(h_2), \]

\[ r_{106} = \cosh(h_2), \]

\[ r_{107} = \cosh(h_2), \]

\[ r_{108} = \cosh(h_2), \]

\[ r_{109} = \cosh(h_2), \]

\[ r_{110} = \cosh(h_2), \]

\[ r_{111} = \cosh(h_2), \]

\[ r_{112} = \cosh(h_2), \]

\[ r_{113} = \cosh(h_2), \]

\[ r_{114} = \cosh(h_2), \]

\[ r_{115} = \cosh(h_2), \]

\[ r_{116} = \cosh(h_2), \]

\[ r_{117} = \cosh(h_2), \]

\[ r_{118} = \cosh(h_2), \]

\[ r_{119} = \cosh(h_2), \]

\[ r_{120} = \cosh(h_2), \]

\[ C_7 = -(r_{112} C_6 + r_{113}), \]

\[ C_6 = r_{110} - r_{113} C_7 - r_{114} C_6, \]

\[ C_5 = r_{112} C_4 + r_{114} C_7 + r_{116} C_5, \]

\[ C_4 = -(r_{105} + r_{110} C_3 + r_{114} C_4), \]

\[ C_3 = -(r_{100} + r_{107} C_4 + r_{118} C_7 + r_{116} C_3), \]

\[ C_2 = r_{114} + C_4, \]

\[ C_1 = -C_3. \]

Case 2:

\[ \begin{align*}
N &= \sqrt{h_1} / 2, \\
S &= \sqrt{1 - \frac{4 \alpha Rv Du^2}{Rc}}, \\
m &= \frac{1}{2Rv}, \\
r_1 &= 1/2N^2, \\
r_2 &= \cosh, \\
r_3 &= \sinh, \\
r_4 &= m^2 \cosh, \\
r_5 &= m^2 \cosh + 2m \sinh m,
\end{align*} \]
\[ r_6 = m^2 \sin \text{nhm}, \quad r_7 = 2mc \cosh m + m^2 \sin \text{nhm}, \]
\[ r_8 = -\left(\frac{1}{h_0} + \frac{1}{Ra} \right) r_5, \quad r_9 = e^{-\eta} \cos N_d, \]
\[ r_{10} = e^{-\eta} \sin N_d, \quad r_{11} = e^{-\eta} \cos N_d, \]
\[ r_{12} = e^{-\eta} \sin N_d, \quad r_{13} = \cosh \text{nhm}, \]
\[ r_{14} = \cosh \text{nhm}, \quad r_{15} = \sinh \text{nhm}, \]
\[ r_{16} = \text{dsinhm}, \]
\[ r_{17} = Ne^{-\eta}(\cos N_d - \sin N_d), \]
\[ r_{18} = Ne^{-\eta}(\cos N_d + \sin N_d), \]
\[ r_{19} = Ne^{-\eta}(\cos N_d + \sin N_d), \]
\[ r_{20} = Ne^{-\eta}(\cos N_d + \sin N_d), \]
\[ r_{21} = m \text{trvmnhm}, \]
\[ r_{22} = R\text{v}(md \text{sinhnhm} + \coshnhm), \]
\[ r_{23} = m \text{trvcoshnm}, \]
\[ r_{24} = R\text{v}(md \text{coshnhm} + \sinhnhm), \]
\[ r_{25} = 2Ne^{-\eta} \sin N_d, \]
\[ r_{26} = -2N^2e^{-\eta} \cos N_d, \]
\[ r_{27} = -2N^2e^{-\eta} \sin N_d, \]
\[ r_{28} = 2Ne^{-\eta} \cos N_d, \]
\[ r_{29} = R\text{v}^2 \cosh \text{nhm} + \frac{1}{D_0} \cosh \text{nhm}, \]
\[ r_{30} = R\text{v}^2 \cosh \text{nhm} + \frac{1}{D_0} \cosh \text{nhm}, \]
\[ r_{31} = -2R\text{v} \text{coshm}, \]
\[ r_{32} = -2R\text{v} \text{coshm}, \]
\[ r_{33} = -2R\text{v} \text{coshm}, \]
\[ r_{34} = -2R\text{v} \text{coshm}, \]
\[ r_{35} = 2N^3e^{-\eta} \sin N_d - \cos N_d, \]
\[ r_{36} = 2N^3e^{-\eta} \sin N_d + \cos N_d, \]
\[ r_{37} = 2N^3e^{-\eta} \sin N_d - \cos N_d, \]
\[ r_{38} = R\text{v}^2 \coshm, \]
\[ r_{39} = R\text{v}^2 \coshm, \]
\[ r_{40} = -2R\text{v} \text{coshm}, \]
\[ r_{41} = R\text{v}^2 \coshm, \]
\[ r_{42} = R\text{v}^2 \coshm, \]
\[ r_{43} = R\text{v}^2 \coshm, \]
\[ r_{44} = R\text{v}^2 \coshm, \]
\[ r_{45} = R\text{v}^2 \coshm, \]
\[ r_{46} = R\text{v}^2 \coshm, \]
\[ r_{47} = R\text{v}^2 \coshm, \]
\[ r_{48} = R\text{v}^2 \coshm, \]
\[ r_{49} = R\text{v}^2 \coshm, \]
\[ r_{50} = R\text{v}^2 \coshm, \]
\[ r_{51} = R\text{v}^2 \coshm, \]
\[ r_{52} = R\text{v}^2 \coshm, \]
\[ r_{53} = R\text{v}^2 \coshm, \]
\[ r_{54} = R\text{v}^2 \coshm, \]
\[ r_{55} = R\text{v}^2 \coshm, \]
\[ r_{56} = R\text{v}^2 \coshm, \]
\[ r_{57} = R\text{v}^2 \coshm, \]
\[ r_{58} = R\text{v}^2 \coshm, \]
\[ r_{59} = R\text{v}^2 \coshm, \]
\[ r_{60} = R\text{v}^2 \coshm, \]
\[ r_{61} = R\text{v}^2 \coshm, \]
\[ r_{62} = R\text{v}^2 \coshm, \]
\[ r_{63} = R\text{v}^2 \coshm, \]
\[ r_{64} = R\text{v}^2 \coshm, \]
\[ r_{65} = R\text{v}^2 \coshm, \]
\[ r_{66} = R\text{v}^2 \coshm, \]
\[ r_{67} = R\text{v}^2 \coshm, \]
\[ r_{68} = R\text{v}^2 \coshm, \]
\[ r_{69} = R\text{v}^2 \coshm, \]
\[ r_{70} = R\text{v}^2 \coshm, \]
\[ r_{71} = R\text{v}^2 \coshm, \]
\[ r_{72} = R\text{v}^2 \coshm, \]
\[ r_{73} = R\text{v}^2 \coshm, \]
\[ r_{74} = R\text{v}^2 \coshm, \]
\[ r_{75} = R\text{v}^2 \coshm, \]
\[ r_{76} = R\text{v}^2 \coshm, \]
\[ r_{77} = R\text{v}^2 \coshm, \]
\[ r_{78} = R\text{v}^2 \coshm, \]
\[ r_{79} = R\text{v}^2 \coshm, \]
\[ r_{80} = R\text{v}^2 \coshm, \]
\[ r_{81} = R\text{v}^2 \coshm, \]
\[ r_{82} = R\text{v}^2 \coshm, \]
\[ r_{83} = R\text{v}^2 \coshm, \]
\[ r_{84} = R\text{v}^2 \coshm, \]
\[ r_{85} = R\text{v}^2 \coshm, \]
\[ r_{86} = R\text{v}^2 \coshm, \]
\[ r_{87} = R\text{v}^2 \coshm, \]
\[ r_{88} = R\text{v}^2 \coshm, \]
\[ r_{89} = R\text{v}^2 \coshm, \]
\[ r_{90} = R\text{v}^2 \coshm, \]
\[ r_{91} = R\text{v}^2 \coshm, \]
\[ r_{92} = R\text{v}^2 \coshm, \]
\[ r_{93} = R\text{v}^2 \coshm, \]
\[ r_{94} = R\text{v}^2 \coshm, \]
\[ r_{95} = R\text{v}^2 \coshm, \]
\[ r_{96} = R\text{v}^2 \coshm, \]
\[ r_{97} = R\text{v}^2 \coshm, \]
\[ r_{98} = R\text{v}^2 \coshm, \]
\[ r_{99} = R\text{v}^2 \coshm, \]
\[ r_{100} = R\text{v}^2 \coshm, \]

Case 3:
\[ R = \sqrt{\frac{\text{Max}(\text{v}, \text{w})^2}{\text{Re}} - 1}, \quad d_1 = \frac{\sqrt{\text{Max}(\text{v}, \text{w})^2}}{2}, \]
\[ d_2 = \sqrt{\frac{1 + \text{Re} - 1}{2}}, \quad \alpha = \frac{d_1}{\sqrt{\text{Max}(\text{v}, \text{w})^2}}, \]
\[ \beta = \frac{d_2}{\sqrt{\text{Max}(\text{v}, \text{w})^2}}, \quad t = \frac{\text{Re}}{\sqrt{2}}, \]
\[ r_1 = \frac{1}{\text{Re}}, \quad r_2 = e^{\alpha} \cos \beta, \]
\[ r_3 = e^{\alpha} \sin \beta, \quad r_4 = e^{\alpha} \cos \beta, \]
\[ r_5 = e^{\alpha} \sin \beta, \quad r_6 = e^{\alpha} \cos \beta, \]
\[ r_7 = e^{\alpha} \sin \beta, \quad r_8 = e^{\alpha} \cos \beta, \]
\[ r_9 = e^{\alpha} \sin \beta, \quad r_{10} = e^{\alpha} \cos \beta, \]
\[ r_{11} = e^{\alpha} \sin \beta, \quad r_{12} = e^{\alpha} \cos \beta, \]
\[ r_{13} = e^{\alpha} \sin \beta, \quad r_{14} = e^{\alpha} \cos \beta, \]
\[ r_{15} = e^{\alpha} \sin \beta, \quad r_{16} = e^{\alpha} \cos \beta, \]
\[ r_{17} = e^{\alpha} \sin \beta, \quad r_{18} = e^{\alpha} \cos \beta, \]
\[ r_{19} = e^{\alpha} \sin \beta, \quad r_{20} = e^{\alpha} \cos \beta, \]

74
\[ r_0 = \frac{e^{-d}}{d_{\text{max}}} \sin \beta - \frac{R}{R_0} e^{-d} \left( (a^2 - \beta^2) \sin \beta - 2a\beta \cos \beta \right), \]
\[ r_1 = \frac{1}{1-H} \frac{R}{R_0}, \]
\[ r_{11} = e^{-Nd} \cos Nd, \]
\[ r_{12} = e^{-Nd} \sin Nd, \]
\[ r_{13} = e^{-Nd} \cos Nd, \]
\[ r_{14} = e^{-Nd} \sin Nd, \]
\[ r_{15} = e^{-Nd} \cos \beta d, \]
\[ r_{16} = e^{-Nd} \sin \beta d, \]
\[ r_{17} = e^{-Nd} \cos \beta d, \]
\[ r_{18} = e^{-Nd} \sin \beta d, \]
\[ r_{19} = \frac{2k^2}{R_0} e^{-Nd} \sin Nd, \]
\[ r_{20} = -\frac{2k^2}{R_0} e^{-Nd} \cos Nd, \]
\[ r_{21} = -\frac{2k^2}{R_0} e^{-Nd} \sin Nd, \]
\[ r_{22} = \frac{2k^2}{R_0} e^{-Nd} \cos Nd, \]
\[ r_{23} = \frac{2k^2}{R_0} e^{-Nd} \sin Nd, \]
\[ r_{24} = \frac{e^a}{D_{\text{max}}} \sin \beta \sin \beta \sin \beta + 2a\beta \cos \beta \sin \beta, \]
\[ r_{25} = -\frac{2k^2}{R_0} e^{-Nd} \cos \beta d, \]
\[ r_{26} = \frac{e^{-Nd}}{D_{\text{max}}} \cos \beta d, \]
\[ r_{27} = N e^{-Nd} \sin Nd + \sin Nd, \]
\[ r_{28} = N e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{29} = -N e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{30} = -N e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{31} = R \cos \beta \sin \beta \sin \beta + a \sin \beta \sin \beta, \]
\[ r_{32} = -R \cos \beta \sin \beta \sin \beta + a \sin \beta \sin \beta, \]
\[ r_{33} = \frac{2k^2}{R_0} e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{34} = \frac{k^2}{R_0} e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{35} = \frac{2k^2}{R_0} e^{-Nd} \cos Nd - \cos Nd - \cos Nd, \]
\[ r_{36} = \frac{2k^2}{R_0} e^{-Nd} \cos Nd - \cos Nd - \cos Nd, \]
\[ r_{37} = \frac{2k^2}{R_0} e^{-Nd} \cos Nd - \cos Nd - \cos Nd, \]
\[ r_{38} = -\frac{2k^2}{R_0} e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{39} = -\frac{2k^2}{R_0} e^{-Nd} \cos Nd + \sin Nd, \]
\[ r_{40} = R e^{-Nd} \left( \frac{1}{1-H} \frac{R}{R_0} \right) \left( (a^2 - 3a\beta^2) \sin \beta + (a^2 - 3a\beta^2) \cos \beta d \right), \]
\[ r_{41} = R e^{-Nd} \left( \frac{1}{1-H} \frac{R}{R_0} \right) \left( (a^2 - 3a\beta^2) \sin \beta + (a^2 - 3a\beta^2) \cos \beta d \right), \]
\[ r_{42} = R e^{-Nd} \left( \frac{1}{1-H} \frac{R}{R_0} \right) \left( (a^2 - 3a\beta^2) \sin \beta + (a^2 - 3a\beta^2) \cos \beta d \right), \]
\[ r_{43} = r_3 \]
\[ r_{44} = r_3 \]
\[ r_{45} = r_3 \]
\[ r_{46} = r_7 - r_0 r_43, \]
\[ r_{47} = r_7 - r_0 r_44, \]
\[ r_{48} = r_7 - r_0 r_45, \]
\[ r_{49} = r_7 - r_0 r_46, \]
\[ r_{50} = r_7 - r_0 r_47, \]
\[ r_{51} = r_7 - r_0 r_48, \]
\[ r_{52} = r_7 - r_0 r_49, \]
\[ r_{53} = r_7 - r_0 r_50, \]
\[ r_{54} = r_7 - r_0 r_51, \]
\[ r_{55} = r_7 - r_0 r_52, \]
\[ r_{56} = r_7 - r_0 r_53, \]
\[ r_{57} = r_7 - r_0 r_54, \]
\[ r_{58} = r_7 - r_0 r_55, \]
\[ r_{59} = r_7 - r_0 r_56, \]
\[ r_{60} = r_7 - r_0 r_57, \]
\[ r_{61} = r_7 - r_0 r_58, \]
\[ r_{62} = r_7 - r_0 r_59, \]
\[ r_{63} = r_7 - r_0 r_60, \]
\[ r_{64} = r_7 - r_0 r_61, \]
\[ r_{65} = r_7 - r_0 r_62, \]
\[ r_{66} = r_7 - r_0 r_63, \]
\[ r_{67} = r_7 - r_0 r_64, \]
\[ r_{68} = r_7 - r_0 r_65, \]
\[ r_{69} = r_7 - r_0 r_66, \]
\[ r_{70} = r_7 - r_0 r_67, \]
\[ r_{71} = r_7 - r_0 r_68, \]
\[ r_{72} = r_7 - r_0 r_69, \]
\[ r_{73} = r_7 - r_0 r_70, \]
\[ r_{74} = r_7 - r_0 r_71, \]
\[ r_{75} = r_7 - r_0 r_72, \]
\[ r_{76} = r_7 - r_0 r_73, \]
\[ r_{77} = r_7 - r_0 r_74, \]
\[ r_{78} = r_7 - r_0 r_75, \]
\[ C_7 = -(r_7 C_8 + r_7), \]
\[ C_6 = r_7 - r_0 C_7 - r_7 C_8, \]
\[ C_5 = -(r_7 C_6 + r_7 C_7 + r_7), \]
\[ C_4 = -(r_7 C_5 + r_7 C_6 + r_7 C_7), \]
\[ C_3 = -(r_7 C_4 + r_7 C_5 + r_7), \]
\[ C_2 = C_4 + r_7, \]
\[ C_1 = -C_3. \]