Mixed Boundary Conditions for Two-Dimensional Transient Heat Transfer Conduction under Lattice Boltzmann Simulations

R. Chaabane, F. Askri and S.B. Nasralla

Laboratoire d'Études des Systèmes Thermiques et Energétiques
Ecole Nationale des Ingénieurs de Monastir
Av. Ibn ElJazzar 5019 Monastir- Tunisie

†Corresponding Author Email: Raoudha.Chaabane@issatgb.rnu.tn

(Received April 25, 2010; accepted March 13, 2011)

ABSTRACT

In this paper, lattice Boltzmann implementations of several types of boundary conditions are introduced and numerically demonstrated. A thermal lattice BGK model is used to simulate thermal fields for flows. The unknown thermal distribution functions at the boundary are subjected to the bounce back concept which is determined consistently with Dirichlet and/or Neumann and/or convective boundary conditions. A consistency analysis using heat transfer conduction is given and the algorithms are numerically tested in two space dimensions with respect to accuracy, numbers of iterations and CPU time. The method is used to simulate conduction transfer problems; numerical results and reference’s solutions are found in satisfactory agreement for thermal fields.

Keywords: Lattice Boltzmann method, Boundary conditions, Dirichlet boundary, Newmann boundary, Convective boundary, 2D heat transfer.

NOMENCLATURE

- \( \varepsilon_i \) propagation speed in the direction \( i \) in the lattice, m/s
- \( \varepsilon_j \) propagation velocity in the direction \( i \) in the lattice, m/s
- \( f_i \) particle distribution function in the \( i \) direction, K
- \( T \) Dimensional temperature, K
- \( f_i^{(0)} \) equilibrium particle distribution function in the \( i \)-direction, K
- \( \tau \) time, s
- \( c_p \) Specific heat, \( m^2s^{-1}K^{-1} \)
- \( \tau \) relaxation time in the LBM, s
- \( c_s \) Lattice sound velocity, \( m/s \)
- \( k \) Thermal conductivity, \( Wm^{-1}K^{-1} \)
- \( \xi \) dimensionless time
- \( \theta \) Non dimensional temperature
- \( x, y \) rectangular coordinates, m
- \( E, W, N, S \) east, west, north, south
- \( \sigma \) Stefan-Boltzman constant
- \( \Delta t \) Time step, s
- \( \Delta x \) Lattice constant, m
- \( \Omega_j \) collision operator
- \( \rho \) density, \( Kg/m^3 \)
- \( \alpha \) Thermal diffusivity, \( m^2s^{-1} \)
- \( \varepsilon \) Dimensional variable

1. INTRODUCTION

Over the last decade the lattice-Boltzmann (LB) methods (Frisch et al. 1987; McNamara and Zanetti 1989; Succi et al. 1991; Benzi et al. 1992; Wolf-Gladrow 2000) have achieved great success as alternative and efficient numerical schemes in the simulation of a variety of transport phenomena in porous media (Chen and Doolen 1998; Spaid and Phelan 1997; Maier et al. 1998; Manz et al. 2004; Maier et al. 1999; Bernsdorf et al. 2000; Clague et al. 2000; Hill et al. 2001; Békri et al. 2001; Drazer and Koplik 2001; Maier et al. 2002; Mishra et al. 2009; Kandhai et al. 2002; Schure et al. 2002; Kang et al. 2002; Zeiser et al. 2002; Békri et al. 2003), heat exchangers, cooling of electronic components, solar collectors, thermal insulation, air heating systems for solar dryers, passive solar heating and storage technology to name just a few (Ho et al. 2002; He et al. 1998; Xi et al. 1999; Takada et al. 2000; Wolf-Gladrow 2000).
A variety of boundary conditions are imposed on the rectangular geometry.

For the problem under consideration, and in the absence of convection and radiation, the energy equation is given by

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + Q$$

(1)

Conventional numerical approaches such as the finite difference and finite element methods are based on the discretization of partial differential equations. In contrast, the LBM is based on the discrete Boltzmann kinetic equation. For heat transport problems, the internal energy evolution equation of the two-dimensional nine-speed (D2Q9) lattice Boltzmann model is given by Qian et al. (1992)

$$\frac{\partial f_i(r,t)}{\partial t} + \bar{v}_i \nabla f_i(r,t) = \Omega_i, \quad i=1,2,3,\ldots,b$$

(2)

The collision operator \(\Omega_i\) represents the rate of change of \(f_i\) due to collisions. It incorporates all the physics and modelling of any particular problem at hand. The simplest model for \(\Omega_i\) is the Bhatnagar–Gross–Krook (BGK) model (Drazer and Koplik 2001)

$$\Omega_i = -\frac{1}{\tau}[f_i(\bar{r},t) - f_i^0(\bar{r},t)]$$

(3)

\(f_i\) is the particle distribution function denoting the number of particles at the lattice node \(\bar{r}\) at time \(t\) moving in direction \(i\) with velocity \(\bar{v}_i\) along the lattice link \(\Delta \bar{r} = \bar{e}_i \Delta t\) connecting the nearest neighbours. \(b\) is the number of directions in a lattice through which the information propagates.

The basis of the discrete velocity model is a finite set of virtual velocities \(\bar{v}_i\) or equivalently, of virtual fluxes of the considered scalar field \(T(\bar{r},t)\) which given by

$$T(\bar{r},t) = \sum_{i=0}^{b} f_i(\bar{r},t) \bar{v}_i$$

(4)

The observed flux is expressed by

$$\sum_{i=0}^{b} f_i(\bar{r},t) \bar{v}_i$$

(5)

The well-known D2Q9 lattice model (Fig.1) will be considered here. In that model, the set of \(\bar{v}_i\)’s is such that they connect the point, on which the lattice stencil is centred, to its nearest neighbours on a spatial grid with uniform spacing in both coordinate directions. Any LBM advances the probability densities \(f_i(r,t)\) in time any combination of these boundary conditions in order to model almost any 2D heat transfer situation subjected to varying boundary conditions.

2. THERMAL LATTICE BOLTZMANN MODEL

The heat transfer in a 2-D rectangular enclosure is considered. Thermo-physical properties of the medium are assumed constant.

The numerical approach that we present in this work can cope with any geometry and, thus, it will be particularly efficient in resolving details of the flow field that govern transport in a transient conduction or/and radiation equation.

The objective of the present work is to compare the performance of the present LBM code in solving transient conduction heat transfer in two-dimensional geometry with LBM code available in Mishra et al. (2009). For this, we consider two-dimensional rectangular geometry where one or two boundaries can be at prescribed heat flux conditions. The energy equation is solved using the LBM and obtained results are compared with reference’s ones Mishra et al. (2009).

Then, we aims to extend the application of the LBM to solve heat conduction problems dealing with temperature as well as heat flux or/and convective boundary conditions.

To that end, we consider two-dimensional rectangular geometry where one boundary is at prescribed heat flux conditions and the remaining ones are subjected to a convective boundary condition.

A second benchmark problem dealing with transient conduction heat transfer in a two dimensional rectangular geometry where the four boundaries are subjected to a convective boundary condition is simulated.

Then, mixed boundary conditions is used showing the flexibility of the method and its efficiency to deal with...
and thereby computes the evolution of the considered scalar. In the absence of external sources or fluxes for the scalar, the corresponding discrete evolution equation can be written in the following general form:

\[
\frac{\partial f_i(\vec{r}, t)}{\partial t} + \vec{e}_i \cdot \vec{V} f_i(\vec{r}, t) = -\frac{1}{\tau} [f_i(\vec{r}, t) - f_i^{(0)}(\vec{r}, t)]
\]  

(6)

Fig.1. The D2Q9 lattice Boltzmann model.

It is a single-relaxation-time model with relaxation constant \( \tau \) that can be related, via Chapman–Enskog analysis, to the diffusivity of the medium. \( f_i^{(0)} \) is the equilibrium distribution function.

The relaxation time can be related with the thermal diffusivity, the lattice velocity \( C \) and the time step \( \Delta t \) (Maier et al. 2002) by the following relation

\[
\tau = \frac{3a}{|C|^2} \frac{\Delta t}{2}
\]  

(7)

For the D2Q9 model in particular, the 9 velocities \( \vec{e}_i \) and their corresponding weights \( w_i \) are the following

\[
\vec{e}_0 = (0, 0) \quad w_0 = 1
\]

(8)

\[
\vec{e}_i = (\cos(\phi_i), \sin(\phi_i)) C \quad \text{for} \quad \phi_i = (i-1)\pi / 2
\]

(9)

\[
\vec{e}_i = \sqrt{2}(\cos(\phi_i), \sin(\phi_i)) C \quad \text{for} \quad \phi_i = (i-5)\pi / 2 + \pi
\]

(10)

\[
w_i = \begin{cases} 
4 & \text{for } i = 1, 2, 3, 4 \\
9 & \text{for } i = 5, 6, 7, 8
\end{cases}
\]  

(11)

(12)

(13)

Where

\[
C = \Delta x / \Delta t = \Delta y / \Delta t \quad \Delta x \text{ and } \Delta t \text{ are the lattice space and the lattice time step size, respectively, which are set to unity. the weights satisfy the relation } \sum_{i=0}^{9} w_i = 1.
\]

After discretization, and considering heat generation, Eq. (6) can be written as

\[
f_i(\vec{r} + \vec{e}_i \Delta t, t + \Delta t) = f_i(\vec{r}, t) - \frac{\Delta t}{\tau} [f_i(\vec{r}, t) - f_i^{(0)}(\vec{r}, t)] + w_i \frac{\Delta t}{\epsilon} Q^*\]

(14)

Where \( Q^* \) is the non dimensional heat generation and \( w_i \) is the weight in corresponding direction.

To process Eq. (8), an equilibrium distribution function is required. For heat conduction problems, this is given by

\[
f_i^{(0)}(\vec{r}, t) = nT(\vec{r}, t)
\]

(15)

For the different sets of boundary condition, the boundaries are based on the properties of the known and unknown populations on each side as shown on Fig. 2.

3. RESULTS AND DISCUSSION

We consider transient heat conduction problems in 2-D Cartesian geometry with a variety of initial and boundary conditions.

Case1: the four boundaries are at known temperatures

The initial and the boundary conditions for case 1 are the following

Initial condition

\[
T(x, y, 0) = T_{ref}
\]  

(16)

Boundary conditions

\[
T(x, 0, t) = 0.25T_{ref} \quad (17)
\]

\[
T(x, Y, t) = T(0, y, t) = T(X, y, t) = T_{ref}
\]

(18)

Steady state conditions were assumed to have been achieved when the temperature difference between two consecutive time levels at each lattice centre did not exceed \( 10^{-4} \). Non dimensional time was defined as \( \xi = \alpha t / L^2 \) where \( L \) is the characteristic length. \( \Delta \xi \) was taken as \( 10^{-4} \).

To check the accuracy of the present LBM algorithm, the same problem was solved using the finite volume method and the results given by the two algorithms are compared with those available in the literature.

In Fig.3, the non dimensional centreline ( \( x/X=0.5 \) ) temperature has been compared at different instants \( \xi \) for the case 1.
Case 2: Effects of heat generation and the four boundaries are at specified temperatures.

In Fig. 4, the effects of volumetric heat generation are shown. The non dimensional volumetric heat generation is taken as unity. Effect of heat generation is very less in the beginning compared to steady state because it takes some time to influence the temperature profile.

For the 2-D geometry, the number of iterations for a 50x50 grid is 3719 (135.95 seconds) compared to that cited at the literature 3257 (Mishra et al. 2009).

The time-space evolution of the isotherms is plotted in Fig. 5 when the four boundaries are at specified temperatures (Dirichlet boundary condition) in the presence of non dimensional heat generation ($Q^T = 2$).

Fig. 3. Centerline ($x/X = 0.5$) temperature evolution for different instants (case 1).

Fig. 4. Comparison of centreline ($x/X = 0.5$) temperature in the presence and the absence of heat generation.

Fig. 5. Isotherms when the four boundaries are at specified temperatures in the presence of non dimensional heat generation ($Q^T = 2$) for different $\xi$. 

$\xi = 0.001$

$\xi = 0.01$

$\xi = 0.05$
**Case3**: The bottom and top boundaries are at prescribed fluxes and remaining two boundaries at known temperatures.

The system is initially at temperature $T_0$. For time $t>0$, the south and the north boundaries are subjected to heat fluxes $q_{s}$ and $q_{n}$, respectively. The east and the west boundaries are kept at temperatures $T_e$ and $T_w$, respectively.

Initial condition

$$T(x, y, 0) = T_0$$

Boundary conditions

$$q(x, 0, t) = q_s$$
$$q(x, Y, t) = q_n$$
$$T(0, y, t) = T(X, y, t) = T_0$$

It is seen from the Fig. 6 that the steady state results match exactly which each other.

![Fig. 6](image1)

**Fig. 6.** Centreline($x/X=0.5$) temperature evolution for different instants (case3).

In Fig. 7, we present the time-space evolution of the isotherms when the bottom and the top boundaries are at prescribed fluxes and remaining two boundaries at known temperatures.

<table>
<thead>
<tr>
<th>Size</th>
<th>Lattices</th>
<th>iterations</th>
<th>CPU time (seconds)</th>
<th>Temperature at steady state($x/X=0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8x8</td>
<td></td>
<td>6251</td>
<td>12.42</td>
<td>0.44170</td>
</tr>
<tr>
<td>12x12</td>
<td></td>
<td>6076</td>
<td>24.33</td>
<td>0.37722</td>
</tr>
<tr>
<td>20x20</td>
<td></td>
<td>6051</td>
<td>53.026</td>
<td>0.34413</td>
</tr>
<tr>
<td>50x50</td>
<td></td>
<td>6199</td>
<td>286.011</td>
<td>0.34493</td>
</tr>
</tbody>
</table>

To have an idea of the number of iterations for the converged solutions and the CPU time, tests were performed with different lattices.

![Fig. 7](image2)

**Fig. 7.** Isotherms when the bottom and the top boundaries are at prescribed fluxes and remaining two boundaries at known temperatures for different $\xi$.
The LBM code was found to take slightly less number of iterations for the little lattices (Table 1). The effect of heat generation on CPU times (second) and number of iterations when all boundaries at known temperatures, was highlighted in Table 2.

Table 2 Effect of heat generation on CPU times (second) and number of iterations of the LBM code (case 3)

<table>
<thead>
<tr>
<th>Lattice size</th>
<th>iterations</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the absence of heat generation</td>
<td>50x50</td>
<td>6199</td>
</tr>
<tr>
<td>In the presence of heat generation</td>
<td>50x50</td>
<td>6317</td>
</tr>
</tbody>
</table>

Case 4:
For the physical problem addressed in this section, the thermal boundary condition at the three side faces, are subjected to convective heat transfer boundary condition:

\[-k \frac{\partial T}{\partial n} = q + h(T - T_\infty)\]  

(21)

where \( h \) is the convective heat transfer coefficient. \( n \) is the direction of outward normal to the surface concerned.

The bottom face is subjected to a heat flux, the temperature boundary condition is:

\[ \frac{\partial T}{\partial y} = q \]  

(22)

The corresponding non-dimensional forms of the boundary conditions obtained from Eqs. (21-22) are formulated as (shown in Fig. 8):

\[-\frac{\partial \theta}{\partial n} = Bi \theta \]  

(23)

\[ \frac{\partial \theta}{\partial y} = 0 \]  

(24)

where \( Bi = h \times l / k \) is the Biot number.

Fig. 8. Physical problem with mixed Newmann and convective boundary conditions.

In Fig. 9, we present the time-space evolution of the isotherms for case 4.
**Case 5:**
A four convective boundary conditions benchmark is studied in this section as presented in Fig. 10. In addition the medium is subjected to a heat generation conditions ($Q^* = 1$).

In order to analyze the effect of the non-dimensional Biot number, steady state non dimensional temperature is plotted for the case of four connectives boundaries as shown in Figs. 11, 12, and 13.

![Fig. 10. Physical problem with convective boundary conditions.](image1)

![Fig. 11. Effect of Biot number on steady state non dimensional temperature, when the four boundaries are subjected to a convective boundary condition along $y/Y$ axis.](image2)

![Fig. 12. Effect of Biot number on steady state nondimensional temperature, when the four boundaries are subjected to a convective boundary condition along $x/X$ axis.](image3)

![Fig. 13. Isotherms when the four boundaries are convective and the medium is subjected to a unity heat generation.](image4)
Case 6:
The mixed convection/conduction/insulated boundary conditions example is constrained as shown in Fig.14. The distributions of isotherms are plotted in the Fig. 15 for different $\xi$. A unity dimensional Biot number is considered. $T_D=T_w=1$ and $T_m=5T_0$

![Fig. 14. Physical problem with mixed dirichlet/Newmann and convective boundary conditions.]

The CPU times and the number of iterations of the LBM code ($Bi = 30$) for a 50x50 grid is presented in the Table 3 for different convective boundary cases.

Table 3 CPU times (second) and number of iterations of the LBM code ($Bi = 30$) for a 50x50 grid

<table>
<thead>
<tr>
<th>iteration</th>
<th>CPU time (s)</th>
<th>Temperature at steady state ($x/X=y/Y=0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 convective boundaries</td>
<td>5126</td>
<td>119.5</td>
</tr>
<tr>
<td>3 convective boundaries $Q_s=0$</td>
<td>7087</td>
<td>148.2</td>
</tr>
<tr>
<td>3 convective boundaries $Q_s=1$</td>
<td>4381</td>
<td>104.9</td>
</tr>
<tr>
<td>2 convective boundaries $Q_s=1$, $Q_n=0$</td>
<td>5260</td>
<td>118.8</td>
</tr>
</tbody>
</table>

5. CONCLUSIONS

The lattice Boltzmann method is used to solve transient heat conduction problems in two dimensional geometries with uniform lattices.

Different sets of mixed boundary conditions were considered namely constant temperature and/or flux boundary conditions and/or convective boundary conditions.

The case of volumetric heat source in the medium was also taken up. In this paper, the problem was analyzed using two numerical approaches, lattice Boltzmann method and the finite volume method.

![Fig. 15. Isotherms when the left the top boundaries are at specified temperatures, the bottom boundary is convective and the south one is insulated.]

$\xi = 0.001$

$\xi = 0.01$

$\xi = 0.05$
Results given by the two numerical approaches are compared with those available in the literature and good agreement is obtained. On the other hand, the effect of lattice size is highlighted via the number of iterations and the CPU time. The considered 2D geometry is a simple one, to allow simple validation. Advection and radiation are omitted. Thus, it remains to demonstrate the viability of the LBM as heat diffusion-advection solver knowing that using LBM will allow us to output the temperature distribution in an extremely simple and accurate way.

REFERENCES


