Chemically Reactive Solute Distribution in a Steady MHD Boundary Layer Flow over a Stretching Surface

M.S. Uddin¹, K. Bhattacharyya², G.C. Layek²† and W.A. Pk³

¹Mathematics Discipline, Khulna University, Khulna-9208, Bangladesh
²Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India
³Department of Mathematics, The University of Rajshahi, Rajshahi-6205, Bangladesh

†Corresponding Author Email: goralayek@yahoo.com
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ABSTRACT

The paper is concerned to find the distribution of the chemically reactant solute in the MHD flow of an electrically conducting viscous incompressible fluid over a stretching surface. The first order chemical reaction and the variable solute distribution along the surface are taken into consideration. The governing partial differential equations along with appropriate boundary conditions for flow field and reactive solute are transformed into a set of non-linear self-similar ordinary differential equations by using scaling group of transformations. An exact analytic solution is obtained for the velocity field. Using this velocity field, we obtain numerical solution for the reactant concentration field. It reveals from the study that the values of concentration profile enhances with the increase of the magnetic field and decreases with increase of Schmidt number as well as the reaction rate parameter. Most importantly, when the solute distribution along the surface increases then the concentration profile decreases.

Keywords: MHD boundary layer, Stretching surface, Chemically reactive solute, Scaling group of transformations.

NOMENCLATURE

- \( a \) stretching constant
- \( B \) magnetic field
- \( C \) concentration
- \( C_0 \) positive solute constant
- \( C_w \) solute distribution along the stretching surface
- \( C_\infty \) constant solute in the free stream
- \( D \) diffusion coefficient
- \( E \) electric field
- \( f \) non-dimensional stream function
- \( \eta \) similarity variable
- \( G \), \( H \) absolute invariants
- \( J \) current density
- \( k \) reaction rate of the solute
- \( M \) magnetic parameter
- \( \nu \) kinematic viscosity
- \( \phi, \psi \) variables
- \( \rho \) density of the fluid
- \( \beta \) reaction rate parameter
- \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \) transformation parameters
- \( \alpha', \alpha'', \beta', \beta'', \beta''' \) transformation parameters
- \( \eta', \eta'' \) variables
- \( R_M \) magnetic Reynolds number
- \( Sc \) Schmidt number
- \( \nu, \psi \) velocity components
- \( \eta \) variable
- \( \Gamma \) scaling group transformations
- \( \alpha \) power-law exponent
- \( n \) a power-law exponent

1. INTRODUCTION

The flows due to stretching sheet in presence of electromagnetic fields are relevant to many practical applications in the metallurgy industries, polymer processing industries, paper production, filaments drawn through a quiescent electrically conducting fluid subject to a magnetic field and the purification of molten metals from nonmetallic inclusions.

The boundary layer equations play a central role in many aspects of fluid mechanics because they describe the motion of a viscous fluid close to a surface. These equations are especially very important since they have the capacity to admit a large number of invariant solutions. Lie-group analysis, also called symmetry analysis was developed by Sophus Lie to find point-transformations that map a given differential equation to itself. This method unifies almost all known exact integration techniques for both ordinary and partial differential equations (Pakdemirli and Yurusoy 1990). Group analysis is the only rigorous mathematical...
method to find all symmetries of a given differential equation and no adhoc assumptions or a prior knowledge of the equation under investigation is needed.

The non-linear character of the partial differential equations governing the motion of the fluid produces difficulties in solving the equations. In fluid mechanics, researchers try to obtain the similarity solutions of the flow equations. In case of scaling group of transformations, the difficulties in solving the equations. In fluid mechanics, equations governing the motion of the fluid produces the non-linear character of the partial differential equation and no adhoc assumptions or a priori knowledge of the equation under investigation is needed.

The scaling group of transformations, the group invariant solutions are nothing but the well known similarity solutions (Mukhopadhyay et al. 2005). A special form of Lie-group of transformations, known as scaling group transformations, is used in this work to find out the full set of symmetries of the flow problem (Mukhopadhyay et al. 2005).

**2. Mathematical Formulation of the Problem**

Consider a steady MHD flow of an electrically conducting viscous incompressible fluid undergoing a first order chemical reaction over a stretching surface. The continuity, momentum and reactive concentration equations for governing the flow and concentration distribution in the boundary layer region along the stretching surface may be written as

\[
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0
\]

(1)

\[
\frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = -\frac{1}{\rho} \left( j \times B \right)_z,
\]

(2)

\[
\frac{\partial}{\partial x} c + v \frac{\partial}{\partial y} c = D \frac{\partial^2 c}{\partial y^2} + k (c - C_\infty),
\]

(3)

where \( u \) and \( v \) are velocity components in \( x \)- and \( y \)-directions respectively, \( v \) is the kinematic viscosity, \( \rho \) is the density of the fluid, \( J \) is the current density and \( B \) is the magnetic field. \( C \) is the concentration, \( D \) is the diffusion coefficient \( k \) denotes the reaction rate of the solute and \( C_\infty \) is constant solute in the free stream. One may note that in writing Eq. (2), we have neglected the induced magnetic field since the magnetic Reynolds number \( R_M \) for the flow is assumed to be small.

The magnetic field \( B \) having components \( (0, B_0, 0) \) with \( B_0 \) non-negative constant, the relation \( \nabla \cdot B = 0 \) is automatically satisfied. It is noted that the electric current in the flow acts parallel to \( z \)-axis (i.e. normal to the plane of the flow). Hence from Ohm’s law we get the components of \( J \) as

\[
j_x = 0, j_y = 0, j_z = \sigma \left[ E_x + (q \cdot B) \right] = \sigma \left[ E_z + u B_0 \right],
\]

(4)

where \( \sigma \) is the constant electrical conductivity of the fluid and \( E_z \) is the component of electric field along the \( z \)-direction and \( q \) is the velocity vector. Now as the flow is steady, Maxwell’s equation gives

\[
\nabla \cdot E = 0
\]

(5)

where \( E \) is the electric field which is along the \( z \)-axis. This gives from Eq. (5) \( \partial E_z / \partial z = 0 \) and \( \partial E_z / \partial x = 0 \) so that \( E_z \) is a function of \( z \) only.

Since the induced magnetic field is neglected in view of the assumption \( R_M < 1 \) electric current in the flow is determined from Ohm's law and not from \( \nabla \times B = \mu_0 J \), \( \mu_0 \) being the magnetic permeability. But the consequence \( \nabla \cdot J = 0 \) of this equation must be satisfied (Shercliff, 1965). This readily gives from Eq. (4), \( E_z = \text{constant} \) since \( E_z \) is independent of \( x \) and \( y \). Thus using Eq. (4), we find from Eq. (2),
In the free stream one can write
\[ \frac{\sigma B_0^2}{\rho} U - \frac{\sigma B_0}{\rho} E_z = 0, \]
which gives \( E_z = -B_0 U \).

Here \( U \) is the free-stream velocity and according to this problem \( U = 0 \).

Finally, the momentum Eq. (6) becomes
\[ \frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = \rho \frac{\partial B_0^2}{\partial y} \frac{\partial u}{\partial y} \]

The appropriate boundary conditions for the velocity components and reactant concentration are given by
\[ u = 0 \text{ at } y = 0 \]
\[ u \to 0 \text{ as } y \to \infty \]
and
\[ C = C_0 + C_\infty x^a \text{ at } y = 0 \]
\[ C \to C_\infty \text{ as } y \to \infty, \]
where \( a \) is assumed to be stretching constant and we consider a variable solute distribution along the stretching surface i.e. \( C = C_0 + C_\infty x^a \), where \( C_0 \) is a positive solute constant, \( n \) is a power-law exponent, which signifies the change of amount of solute in the \( x \)-direction.

Introducing the stream function to this boundary layer flow we get the following relation as
\[ \psi = \frac{\partial \psi}{\partial x} \]

and the concentration \( C \) is related by
\[ C = C_\infty \frac{x^a}{\partial y} \frac{\partial C}{\partial y} = \frac{x^a}{\partial y} \frac{\partial C}{\partial y} \]

and the boundary conditions become
\[ \frac{\partial \psi}{\partial y} = ax \text{ and } \frac{\partial \psi}{\partial x} = 0 \text{ at } y = 0 \]
\[ \frac{\partial \psi}{\partial y} \to 0 \text{ at } y \to \infty \]
\[ C = 1 \text{ at } y = 0 \]
\[ C \to 0 \text{ at } y \to \infty. \]

### 3. INVARIANT SOLUTION THROUGH SCALING GROUP OF TRANSFORMATIONS

We now introduce the simplified form of Lie-group transformations, namely, the scaling group of transformations (Mukhopadhyay et al. 2005) as
\[ \Gamma: \left\{ \begin{array}{l} x' = x e^{\alpha a}, y' = y e^{\alpha a}, \psi' = \psi e^{\alpha}, \\ u' = u e^{\alpha a}, v' = v e^{\alpha} \end{array} \right \} \]

The transformation (16) may be considered as a point transformation, which transformed the coordinates \((x, y, \psi, u, v)\) to the coordinates \((x', y', \psi', u', v')\).

Taking the relations (16) in to account in Eqs. (12) and (13), we obtain respectively
\[ e^{(\alpha a)} \frac{\partial \psi'}{\partial y'} = \frac{\partial \psi'}{\partial y'} - \frac{\partial \psi'}{\partial y'} e^{\alpha a} \]

and
\[ e^{(\alpha a)} x \frac{\partial \psi'}{\partial y'} = e^{(\alpha a)} x \frac{\partial \psi'}{\partial y'} e^{\alpha a} \]

In order that, the system will remain invariant under the group of transformation \( \Gamma \) we then would have the following relations among the transformation parameters
\[ \alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 \]

From (19) we can obtain easily \( \alpha_1 = 0 \) and \( \alpha_3 = \alpha_3 \). The relation \( u' = \frac{\partial \psi'}{\partial y'} \) and \( v' = \frac{\partial \psi'}{\partial x'} \) gives us \( \alpha_1 = \alpha_3 \), \( \alpha_2 = 0 \). In view of these, the boundary conditions (14) and (15) are transformed to
\[ \left\{ \begin{array}{l} \frac{\partial \psi'}{\partial y'} \to 0 \text{ at } y' = 0 \\ C' \to 0 \text{ at } y' \to \infty \end{array} \right \} \]

where the boundary condition \( C' = 1 \) gives \( \alpha_2 = 0 \). Thus the set \( \Gamma \) finally reduces to a one-parameter group transformation
\[ \Gamma: \left\{ \begin{array}{l} x' = x e^{\alpha a}, y' = y, \psi' = \psi e^{\alpha}, \\ u' = u e^{\alpha a}, v' = v \end{array} \right \} \]
Firstly, we consider the absolute invariant, \( \eta \) which is a function of the independent variables and is taken as \( \eta = x' x'' \).

Since the quantity \( \eta \) is absolute invariant, we get \( y' x'' = x' y'' \). Now, \( y' x'' = y' x'' e^{\alpha z} \) if \( s = 0 \) (since \( \alpha \) cannot be 0). Hence, we get the first absolute invariant as \( \eta = y'' \).

We now find the second absolute invariant, \( G = f(\eta) \) which involves the dependent variable \( y' \) and assume that \( G = x' y'' \). Since \( G \) is an absolute invariant, we will find \( r \) such that \( x'' y' = x' y'' \). Now, \( x'' y' = (x' e^{\alpha m}) y' e^{\alpha m} = (x' e^{\alpha m}) y e^{\alpha m} = e^{\alpha (s+1)} x' y' = x' y'' \) if \( r = -1 \). Putting \( r = -1 \), the second absolute invariant \( G \) becomes \( G = x' y'' \), i.e. \( f(\eta) = x' y'' \).

Lastly, we want to find the third absolute invariant, \( H = \phi(\eta) \) which involves the independent variables and the dependent variable \( C' \) and is taken as \( H = x'' C' \). \( H \) is an absolute invariant if \( x'' C' = x x' C' \).

Now, \( x'' C' = (x' e^{\alpha m}) C = (x' e^{\alpha m}) \overline{C} = e^{\alpha (s+1)} x' C = x' C \) if \( p = 0 \). Thus, the third absolute invariant is \( H = C' \) i.e. \( \phi(\eta) = C' \).

Finally, from the absolute invariants, we get the transformations as given below:

\[ \eta = y'' \text{ and } C' = \phi(\eta) \] (23)

In view of the above relations, the Eqs. (17) and (18) become

\[ u f'' + f f'' - f'' = -\frac{\sigma B^2}{\rho} f' = 0 \] (24)

\[ D \phi'' + f \phi' - n f' \phi - k \phi = 0 \] (25)

and the boundary conditions reduced to

\[ f(\eta) = 0, \quad f'(\eta) = a \text{ at } \eta = 0 \] (26)

\[ f'(\eta) \to 0 \text{ as } \eta \to \infty \]

\[ \phi = 1 \text{ at } \eta = 0 \]

\[ \phi \to 0 \text{ as } \eta \to \infty \] (27)

Again, we introduce the following transformations for \( \eta, f \) and \( \phi \) in Eqs. (24)-(27):

\[ \eta = u \alpha \beta \phi, \quad f = u \alpha \beta \phi \quad \text{ and } \quad \phi = u \alpha \beta \phi \] (28)

and we obtain \( \alpha = \alpha', \beta = \beta', \beta' = -\frac{1}{2} \) and \( \alpha'' = \beta'' = 0 \).

Finally, in view of the above transformations and taking \( \phi = \eta, \quad \overline{f} = f \) and \( \overline{\phi} = \phi \), the Eqs. (24) and (25) reduce to the following forms:

\[ f'' + f f'' - f'' = 0 \] (29)

\[ \phi'' + Sc \ f \phi' - Sc(n f' + \beta) \phi = 0 \] (30)

where \( M = \frac{c B^2}{\rho D} \) is the magnetic parameter, \( Sc = \frac{u D}{D} \) is the Schmidt number and \( \beta = k/a \) is the reaction rate parameter of the solute.

The boundary conditions (26) and (27) reduce to the following forms:

\[ f(\eta) = 0, \quad f'(\eta) = 1 \text{ at } \eta = 0 \] (31)

\[ f'(\eta) \to 0 \text{ as } \eta \to \infty \]

\[ \phi(\eta) = 1 \text{ at } \eta = 0 \]

\[ \phi(\eta) \to 0 \text{ as } \eta \to \infty \] (32)

4. SOLUTION OF THE PROBLEM

The Eq. (29) along with the boundary condition (31) is solved analytically (Sarpkaya 1961) and the exact solution is given by

\[ f(\eta) = \frac{1 - \exp \left( -\sqrt{1 + M} \eta \right)}{\sqrt{1 + M}}, \eta \geq 0. \] (33)

After substituting \( \alpha \) of the function \( f \) and using finite-difference technique in the Eq. (30) along with the boundary conditions (32) is solved numerically. The expression for wall shear stress is given by

\[ \left| f'(0) \right| = \sqrt{1 + M} \] which increases with the increase of magnetic field \( M \) and consequently, the boundary layer thickness of the stretching surface decreases.

5. RESULTS AND DISCUSSIONS

The analytic solution of velocity has presented for various values of magnetic parameter \( M \). The reactant solute equation is solved numerically and the results are shown graphically.

The velocity profiles for various values of the magnetic parameter \( M \) have been plotted in Fig. 1. From the figure it is noted that with increase of \( M \), the velocity for any fixed value of \( \eta \) decreases. Thus it is clear that the magnetic field opposes motion. This is due to the fact that variation of \( M \) leads to the variation of Lorenz force producing more resistance to the transport process. Consequently, the moment boundary layer thickness reduces with the increase in \( M \) and this fact is also seen from wall shear stress behaviour.

![Fig. 1. Velocity profiles \( f(\eta) \) for various values of \( M \) vs. \( \eta \).](image-url)
In order to assure the accuracy of applied numerical method, we have compared our obtained results for concentration gradient at the surface $-\phi'(0)$ which is related with the Local Sherwood number by the relation $	ext{Sh}/\text{Re}^{1/2} = -\phi'(0)$ that of Takhar et al. (2000) and Andersson et al. (1994) for $n=0$ (i.e., with constant solute along the surface) in Table 1 and found in excellent agreement.

Table 1 Comparison of the values of $-\phi'(0)$ with that of Takhar et al. (2000) and Andersson et al. (1994) for $n=0$.

<table>
<thead>
<tr>
<th>Sc</th>
<th>$\beta$</th>
<th>Present Study</th>
<th>Takhar et al. (2000)</th>
<th>Andersson et al. (1994)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.15057</td>
<td>0.15042</td>
<td>0.149</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>0.66873</td>
<td>0.67044</td>
<td>0.669</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.17679</td>
<td>1.17761</td>
<td>1.177</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>3.87347</td>
<td>3.87469</td>
<td>3.880</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10.24355</td>
<td>10.24283</td>
<td>10.25</td>
</tr>
</tbody>
</table>

Figure 2 exhibits concentration profiles for various values of $M$. The value of contaminate solute at particular value of $\eta$ increases with the increase of the magnetic parameter $M$ and also the concentration boundary layer thickness increases. This implies that the magnetic field acts to enhance the distribution of the reaction solute on the stretching surface in case of an electrically conducting fluid subject to magnetic field. This result may be useful, in the situation where the enhancement of solute transfer from the surface is the prime important.

![Fig. 2. Concentration profiles $\phi(\eta)$ for various values of $M$.](image)

Now, we concentrate on variation the solute curves for different values of Schmidt number $Sc$. The curves are drawn in the Fig. 3. The Schmidt number has major effects on the distribution of solute. The concentration boundary layer thickness as well as the concentration at a fixed point decreases quickly with increasing values of $Sc$. This is due to the fact that the rate of solute transfer from the surface increases when the Schmidt number increases.

![Fig. 3. Concentration profiles $\phi(\eta)$ for various values of $Sc$.](image)

Figure 4 is the graphical representation of concentration profiles for various values of reaction rate parameter $\beta$. It has been found that the reaction rate parameter affect the solute profiles in similar way as that of the Schmidt number i.e., the increase of $\beta$ reduces both the solute boundary layer thickness and value of the solute at fixed $\eta$. So, in case of the distribution of reactive solute, the reaction rate parameter is a decelerating agent.

![Fig. 4. Concentration profiles $\phi(\eta)$ for various values of $\beta$.](image)

Finally, Figs. 5 and 6 exhibit the concentration profiles in the boundary layer flow region for different values of power-law exponent $n$. It is noticed from Fig. 5 that for the increasing values of $n$ with $n>0$, the curve representing the distribution of solute for specific value of $\eta$ decreases.

![Fig. 5. Concentration profiles $\phi(\eta)$ for various values of $n(\geq 0)$.](image)

![Fig. 6. Concentration profiles $\phi(\eta)$ for various values of $n(<0)$.](image)
While, in Fig. 6 the concentration profile increases with increase in the magnitude of \( n \) with \( n>0 \) and for large negative values of \( n \), the overshoot of solute is observed near the surface. Thus, the effect of increase of \( n \) when the surface concentration is \( C_w = C_\infty + C_0/x^n \) is completely opposite to the effect of increase of \( n \) when the surface concentration is \( C_w = C_\infty + C_0/x^n \). Note that, the wall concentration is constant when \( n=0 \).

6. CONCLUSIONS

In this investigation, an analysis is made to find the behaviour of the distribution of reactive solute undergo a first order reaction in steady MHD boundary layer flow of an electrically conducting incompressible fluid over a stretching surface taking variable surface concentration. Using the scaling group of transformation a set of self-similar equations is obtained from the governing equations. The analytical solution is found for the momentum equation and the equation of reactive solute is solved numerically. The results show that the magnetic field tends to reduce the rate of flow from the wall and is broadening the solute layer. The Schmidt number and the reaction rate parameter reduce the solute boundary layer thickness. Most, importantly, the effects of initial variable solute distribution over a stretching surface is interesting i.e. for the increase in magnitude of \( n \), the concentration decreases when \( n>0 \) whereas increases when \( n<0 \).

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REFERENCES


