On Dissipative Radiative MHD Boundary Layer Flow in a Porous Medium Over a Non Isothermal Stretching Sheet

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ABSTRACT

The paper aims at investigating the effects of Ohmic and viscous dissipations on the steady two-dimensional radiative boundary-layer flow of an incompressible, viscous, electrically conducting fluid caused by a linearly stretching sheet placed at the bottom of fluid saturated porous medium in the presence of uniform transverse magnetic field. The radiative heat flux is assumed to follow Rosseland approximation. The governing system of partial differential equations are converted to ordinary differential equations by using the similarity transformations, which are then solved numerically using shooting method with fourth order Runge-Kutta scheme. The dimensionless temperature distribution is computed for different thermo-physical parameters and presented graphically. The temperature gradient at the sheet and skin friction coefficient are derived numerically and presented through graphs.

Keywords: Numerical study, Boundary layer, Stretching sheet, Porous medium, Dissipation, Thermal radiation, MHD, Convective heat transfer.

NOMENCLATURE

Bo magnetic field intensity
c stretching sheet parameter
cp specific heat at constant pressure
D constant
Ec Eckeret number
k* mean absorption coefficient
K permeability of porous medium
\ell characteristic length
M Hartmann number
N Radiation parameter
Pr Prandtl number
q radiation heat flux
T fluid temperature
T∞ uniform temperature of the ambient fluid

Tw wall temperature
u, v velocity components
x, y Cartesian coordinates

Greek Symbols

ψ stream function
η similarity parameter
λ permeability parameter
κ uniform thermal conductivity
ν kinematic viscosity
ρ density of fluid
σ* Stephan Boltzman constant
σ electrical conductivity
μ coefficient of viscosity
θ dimensionless temperature

1. INTRODUCTION

Viscous dissipation amounts to local production of thermal energy through the mechanism of viscous stresses. This effect is encountered in both the viscous flow of clear fluids and the fluid flow within the porous medium. Although viscous dissipation effect is considered as a “weak” effect when compared with its other counterpart effects but in many situations it has to be reckoned with. The viscous heating aspects in fluids were investigated for its practical interest in polymer industry and the problem was invoked to explain some rheological behavior of silicate melts. The flow studies with viscous heating aspects of viscous fluids demonstrating temperature dependent properties are of immense significance in basic sciences and in contemporary industrial technology such as tribology, instrumentation, food processing, lubrication, polymer manufacturing and many others. Gebhart (1962) came out with observations that devices which operate at high rotational speeds or which are subject to large decelerations experience significant viscous dissipation effect. The effect is felt prominently in strong
gravitational fields and in processes wherein the scale of
the process is very large, e.g. on larger planets, in
large masses of gas in space and in geological processes
in fluids contained in various bodies. It is pertinent to
record that even if viscous dissipation effect is
quantitatively negligible in some cases its qualitative
effect is significantly observed. The works of Gebhart
(1962), Gebhart et al. (1969), Nield (2000) and Magyari
et al. (2003), Rees et al. (2003) shed a light on the
importance of viscous heating.

In recent years, many researchers have shown interest
in the study of boundary-layer flows of viscous fluids
over a stretching sheet simply because boundary-layer
behavior on a moving continuous solid surface is an
important type of flow encountered in several
engineering processes. An example of a moving
continuous surface is a long thread travelling between a
feed roll and a wind-up roll, or a polymer sheet or
filament extruded continuously from a die. Investigation of transport processes in porous media
have been undertaken with utmost interest due to wide
array of applications cutting across different realms
such as geothermal engineering, underground disposal
of nuclear waste, chemical reactors, heat exchangers
and more. Pop et al. (2001), Nield et al. (2006) have
presented relevant comprehensive texts.

Sakiadis (1961) was the first to study the flow in the
boundary-layer on a continuous solid surface. He
considered the boundary layer flow over a flat surface
moving with a constant velocity and formulated a
boundary layer equation for two-dimensional,
axisymmetric flows. Due to entrainment of ambient
fluid, this phenomenon represents a different type of
boundary-layer problem having solution substantially
different from that of boundary-layer flow over semi-
infinite flat plate. Crane (1970) extended the work of
Sakiadis by considering a moving strip, the velocity of
which is proportional to the distance from the slit and
obtained closed form exponential solution. Subsequently, many investigators taking the advantage
of simplicity of geometry and its exact solution
attempted the problem with variety of assumptions.
Gupta and Gupta (1977) investigated the heat and mass
transfer over an isothermal stretching sheet with suction
or blowing. Arunachalam et al. (1978) considered the
thermal boundary-layer in liquid metals with variable
thermal conductivity for a class of flow where the
potential velocity is a power of the distance along a
stationary wall. Grubka et al. (1985) analyzed the
phenomena with prescribed wall temperature and also
with prescribed heat flux and presented their solutions
in terms of Kummer’s function. Chen et al. (1988)
investigated the problem for visco-elastic fluid of
Walter’s liquid B model subject to power law heat flux.
They also obtained their solutions in terms of
Kummer’s function. Vajravelu et al. (1993) addressed
the problem taking the effects of viscous dissipation
and internal heat generation into account. Chauhan et
al. (1995) examined heat transfer in MHD viscous flow
due to stretching of boundary in the presence of
naturally permeable bed. Anderson et al. (1998)
examined the flow of viscous ferro-fluid over a
stretching sheet in the presence of a magnetic dipole to
explore the effects of the magneto-thermo-mechanical
interaction on skin friction and heat transfer. The
problem of linearly stretching sheet was generalized to
one that stretches with a power-law velocity by Afzal et
(1982) investigated the flow of micropolar fluid over a
stretching sheet. Chiam (1995) also investigated the
case of a sheet stretching with a power-law velocity and
having a variable magnetic field of a special form.
Bourtos et al. (2006) studied two-dimensional
boundary-layer stagnation point flow towards a heated
stretching sheet placed in a porous medium using Lie
group method. All the above reported investigations
were limited to flow and heat transfer without taking
radiation into account.

Heat transfer together with radiation from the outer
surface of a heated body embedded in a fluid-saturated
porous medium finds several applications in geophysics
and engineering. Cheng et al. (1977) were the first to
present an analysis for the natural convection flows
about a heated impermeable surface embedded in fluid-
saturated porous media to model the heating of ground
water in an aquifer by dike. However, thermal radiation
at high temperatures significantly affects the heat
transfer and the temperature distribution in the
boundary-layer flow of participating fluid. In fact,
depending on the surface properties and geometry of
the solid, the radiation has dominant impact on flow
and heat transfer in porous media. The effect of
radiation in thermal regime in porous medium has got
wide applications, such as, waste heat storage in
aquifers, gasification of oil shale which is of interest on
combined convection since fluid is pumped into porous
region. In fact, in the case of gasification, large
temperature gradient exists in the vicinity of the
combustion regime making radiation effect dominant.
Raptis (1998) studied radiative micropolar fluid flow
past a continuously moving plate. The radiative heat
transfer studies are very important in space technology
and high temperature processes. But, unfortunately very
little has been reported about the effects of radiation on
the boundary-layer in porous media. However it is
interesting to record that porous medium absorbs/emits
radiation that is transferred to or from a fluid. The fluid
can be regarded to be transparent to radiation, because
the dimensions for radiative transfer among the
solid elements of porous structure are much less than
the radiative mean free path for scattering or absorption
in the fluid (Howell(2000)). Furthermore, it is pertinent
to record that contrary to conduction and convection
heat transfer by thermal radiation is a complex
phenomenon to account for. Actually fluid radiation
studies are confronted with a few difficulties making
the things complex and cumbersome. In radiative heat
transfer, the prediction of fluid absorption is a tedious
task because the radiation is absorbed/emitted not only
at system boundaries but also in the interior of the
system. Furthermore, the absorption coefficients of the
absorbing/emitting fluids are, in general, strongly
dependent on wavelength. The computational procedure
gets difficult with the presence of radiation term in
the energy equation making the equation highly non linear.
In view of all these challenges in radiative studies, the
effect of radiation on convective flow has been
undertaken with reasonable simplifications. Excellent
literature on radiation is available in the well presented
texts by Sparrow et al. (1970), Özisik (1973) and

Magnetohconvection is of considerable interest owing to its frequent occurrence in various realms of industrial technology and geothermal applications, liquid metal fluids, MHD power generation system, high temperature plasmas applicable to nuclear fusion energy conversion. MHD has also been found very useful in controlling boundary-layer transition in the case of subsonic and supersonic flow of gases. Duwairi et al. (2006) presented numerical investigations of magnetohydrodynamic natural convection heat transfer from radiative vertical porous surface. Sunetha et al. (2011) investigated thermal radiation effects on MHD flow past an impulsively started vertical plate in the presence of heat source/sink by taking viscous dissipation into account.

In this communication we wish to study the MHD boundary-layer flow in a porous medium with radiative and dissipative effects. A suitable numerical technique is used to solve the nonlinear energy equation.

2. MATHEMATICAL ANALYSIS

Let us consider the steady, dissipative, MHD twodimensional boundary-layer flow over a linearly stretching sheet placed at the bottom of the fluid saturated porous medium in the presence of radiation. A Cartesian system is used with x-axis is along the sheet and y-axis is normal to it. The sheet is stretched linearly by applying two equal and opposite horizontal forces so that the position of the origin is unaltered. The fluid is considered to be viscous, incompressible, electrically conducting, Newtonian, optically dense and without phase change. We assume that the wall temperature \( T_w > T_e \) where \( T_e \) is the uniform temperature of the ambient fluid. It is assumed that both the fluid and porous medium are in local thermal equilibrium and both the fluid and the surroundings are maintained at a constant temperature far away from the sheet. The radiative heat flux in the energy equation is approximated by Rosseland approximation. A magnetic field of strength \( B_0 \) is also applied transverse to the sheet. The induced magnetic field, the external electric field and the electric field due to polarization of charges are neglected while the Ohmic and viscous dissipations are taken into account.

2.1 Governing Equations and Boundary Conditions

The governing boundary layer equations can be written as

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= u - \frac{\sigma B_0^2 u}{\rho} \\
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \frac{\partial^2 (\frac{\partial u}{\partial y})^2}{\partial y^2}
\end{align*}
\]

The corresponding boundary conditions are

\[
\begin{align*}
y = 0: \quad u &= cx, \quad v = 0 \\
y = 0: \quad T &= T_e(x) = T_0 + \frac{\nabla T}{\varepsilon'} \\
y \to \infty: \quad u &= 0; \quad T \to T_e
\end{align*}
\]

Where the radiation heat flux (Brewster(1972)) is

\[
q_r = -4\sigma^* \frac{\partial T^4}{\partial y} \frac{3k^*}{3k^*}
\]

Here the temperature difference within the flow is assumed to be sufficiently small so that \( T^4 \) may be expressed as a linear function of temperature \( T \), using a truncated Taylor series about the free stream temperature \( T_w \) to yield

\[
T^4 \approx 4T_w^4 T - 3T_w^4
\]

2.2 Similarity Transformation

Let us introduce the following similarity transformations

\[
25
\]
where \( \psi \) is the stream function such that
\[
\begin{align*}
\psi &= \sqrt{c y f(\eta)}, \quad \eta = \sqrt{\frac{c}{y}} \sqrt{\frac{T - T_0}{T_s - T_0}},
\end{align*}
\]
(7)
where \( \psi \) is the stream function such that
\[
\begin{align*}
u = \frac{\partial \psi}{\partial \xi}, \quad \nu = -\frac{\partial \psi}{\partial \xi}
\end{align*}
\]
(8)
From Eq. (7) and Eq. (8) we get velocity components as
\[
\begin{align*}
u = cx f(\eta) \quad \text{and} \quad \nu = -\sqrt{y c f(\eta)}
\end{align*}
\]
(9)
We see that the equation of continuity in view of Eq. (9), is identically satisfied.
The central objective of this paper is to investigate thermal aspects of the problem. In order to solve the energy Eq. (3) the temperature distribution is assumed in the form of similar solution as
\[
\begin{align*}
T &= T_s + D_{-\eta} \bar{\theta}(\eta)
\end{align*}
\]
(10)
The momentum Eq. (2) and energy Eq. (3) in view of Eqs. (5)- (10) reduce to the following form
\[
\begin{align*}
f'' + f f' + f^2 - \left(\frac{1}{\lambda} + M^2\right) f' &= 0
\end{align*}
\]
(11)
\[
\begin{align*}
\frac{1}{Pr}\left(4N\right) f'' + f f' - 2f^2 + \theta M^2 E_{cf} f' + E_{cf} f^2 &= 0
\end{align*}
\]
(12)
with the corresponding boundary conditions
\[
\begin{align*}
\eta &= 0, \quad f = 0, \quad f' = 1, \quad \theta = 1
\end{align*}
\]
(13)
\[
\begin{align*}
\eta \rightarrow \infty, \quad f' = 0, \quad \theta = 0
\end{align*}
\]
where prime denotes differentiation with respect to \( \eta \).
Here \( M \) is the Hartmann number, where \( M^2 = \frac{\sigma B_0^2}{\rho c_y} \),
\[
N = \frac{4 \sigma T^4}{k\kappa}
\]
is the radiation parameter and \( \Pr = \frac{\rho u c_p}{\kappa} \)
is the Prandtl number. Also \( E_{cf} = \frac{c_D}{c_p} \) is the Eckert number and \( \lambda = \frac{c K}{u} \) is the Permeability parameter.

### 2.3 Solution Procedure

The solution of eq. (11) is assumed to be of the form
\[
\begin{align*}
f(\eta) &= A + B e^{-s \eta}
\end{align*}
\]
(14)
where the constants \( A, B \) and \( s \) are given by
\[
\begin{align*}
A &= \frac{1}{s}, \quad B = -\frac{1}{s}, \quad \text{and} \quad s = \sqrt{1+\left(\frac{1}{\lambda} + M^2\right)}
\end{align*}
\]
(15)
Thus the exact solution of Eq. (9) can be written as
\[
\begin{align*}
f(\eta) &= \frac{1}{s}(1 - e^{-s \eta}),
\end{align*}
\]
(16)
\[
\begin{align*}
f'(\eta) &= e^{-s \eta}
\end{align*}
\]
(17)
Thus the exact solution of
\[
\begin{align*}
\bar{\theta}(\eta) &= \frac{1}{s}(1 - e^{-s \eta}),
\end{align*}
\]
(16)
\[
\begin{align*}
\bar{\theta}'(\eta) &= e^{-s \eta}
\end{align*}
\]
(17)
The non-linear boundary value problem given by Eq. (12) with Eqs. (16)-(17) and boundary conditions given by Eq. (13) does not possess closed form analytical solution. Therefore it has been solved numerically by fourth order Runge-Kutta scheme together with shooting method. The computational procedure involved two challenges, firstly determination of \( \eta \) i.e. maximum value of \( \eta \) for which \( \theta(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \). Secondly, in order to employ shooting method, appropriate guesses of \( \theta'(0) \) are required so that the condition at the other end is satisfied.
The secant method of iteration was used to search for missing \( \theta'(0) \). The values of \( \eta_0 \) for which \( \theta(\eta) \) decays exponentially to zero for different set of values of parameters \( \lambda, M, N, E_{cf} \) and \( \Pr \) was chosen after some preliminary investigation. A grid independence study was carried out to examine the effect of the step size \( \Delta \eta \) and the edge of the boundary layer \( \eta \) on the solution in the quest for their optimization. The \( \eta_{max} \) i.e. \( \eta \) at \( \infty \) was so chosen that further changes in it showed little changes (constant till \( 10^8 \) in the values of \( \theta'(0) \) vis a vis boundary condition \( \theta(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \) is satisfied.
A step size \( \Delta \eta = 0.025 \) was found to be sufficiently large for convergence criterion of \( 10^{-6} \). We see that the equation of continuity in view of Eq. (2) is identically satisfied.

The procedure adopted above provides wall temperature gradient at the wall numerically that has been shown graphically.

### 3. SKIN FRICTION

The skin friction \( \tau^* \) at the wall in the non-dimensional form is given as:
\[
\begin{align*}
\tau^* &= \frac{\mu c u}{\sqrt{\nu}} (c')(0) = -s
\end{align*}
\]
(18)
4. RESULTS AND DISCUSSION

In order to get the insight of the phenomenon under study, the profiles for the temperature $\theta(\eta)$ and the wall temperature gradient $\theta'(0)$ have been drawn for variable parameter values. Numerous set of values of the parameters central to the study were used to draw profiles, however, selective cases have been reported. The variations of $\theta(\eta)$ versus $\eta$ have been depicted in Fig. 2 - Fig. 4.

**Figure 2** displays variation in $\theta(\eta)$ for different values of permeability parameter $\lambda$ and Hartmann number $M$. The figure envisages that $\theta(\eta)$ registers increment with the decreasing values of $\lambda$. Here it is worth to note that...
the permeability parameter $\lambda$ is the inverse of Darcian drag force in the porous medium. Thus, larger values of $\lambda$ signifies rather low resistance by the porous medium to cause more ease for fluid trasversal in porous medium. Hence, for larger values of $\lambda$, the thermal boundary layer thickness decays with a physical significance of shorter cooling time. Fig. 2 further reveals that the temperature $\theta(\eta)$ increases with the increasing values of $M$. Actually, the Hartmann number $M$ is the measure of the relative importance of Lorentz force to the viscous hydrodynamic force. Thus, larger values of $M$ indicate stronger magnetic field strength. Furthermore, it is to recall that we have applied transverse magnetic field which retards the flow. This results in hampering of the convective flow with the increasing values of $M$ and consequently leads to thickening of the thermal boundary layer. In order to understand the effect of Hartmann number (magnetic field parameter), Eq.(2) and Eq.(3) require attention.

The term $-\frac{\sigma B_0 u}{\rho}$ in Eq. (2) is the Lorentz force which would act in opposite direction of the flow, hence retards the flow. Further the term $-\frac{\sigma B_0 u}{\rho c_p}$ in Eq. (3) represents Ohmic heating due to electromagnetic work and serves as heat source to cause rise in the fluid temperature. Here it is not out of place to emphasize that the Lorentz force can be used to control the flow of electrically conducting fluid that can serve to assist or oppose the flow (opposing flow and aiding flow).

Figure 3 demonstrates the variation in $\theta(\eta)$ with Prandtl number $Pr$ and the radiation parameter $N$. From the very figure we observe that $\theta(\eta)$ registers decrement with increasing values of $Pr$. Prandtl number physically, signifies the relative importance of momentum diffusion to thermal diffusion in the flow field. In fact dense fluids such as oils enjoy larger Prandtl number values whereas low density fluids such as liquid metals are characterized by low Prandtl number values less than unity. In thermal regime with low Prandtl number (e.g. mercury, $Pr = 0.023$), the thermal diffusivity has an upper hand i.e. the heat diffuses at faster rate compared to the momentum. However it should be noted that low Prandtl number fluids exhibit temperature dependent thermal conductivity. On the contrary to it, fossil oil having low thermal conductivity (high Pr values) gives rise to thinner boundary layer. From the fig.3 it is also observed that $\theta(\eta)$ increases with the increasing values of $N$. The radiation parameter $N$ being reciprocal of Stark number (also known as Stephan number) is the measure of relative importance of thermal radiation transfer to the conduction heat transfer. Thus larger values of $N$ sound dominance of thermal radiation over conduction. Consequently larger values of $N$ are indicative of larger amount of radiative heat energy being poured into the system, causing rise in $\theta(\eta)$.

Figure 4 depicts the effect of Eckert number $Ec$ on the non-dimensional temperature. It is revealed that $\theta(\eta)$ scores growth with the increasing values of $Ec$. Eckert number, physically, is a measure of frictional heat in the system. Hence the thermal regime with larger $Ec$ values is subjected to rather more frictional heating causing a source of rise in the temperature. To be specific, the Eckert number, $Ec$ signifies the relative importance of viscous heating to thermal diffusion. Viscous heating may serve as energy source to modify the temperature regime qualitatively. Here a comparison of Fig. 2 and Fig. 4 is interesting. Close examination of the profiles in these figures would reveal the impact of Eckert number, $Ec$. In Fig. 4 we find that the temperature in the vicinity of the sheet rises considerably (note that at wall $\theta(0)=1$) above 1 for larger Ec values. This is due to the frictional heating.

The variation of rate of heat transfer at the sheet ($-\theta'(0)$) with respect to Prandtl number $Pr$ is shown from Fig. 5 – Fig.7.

![Fig. 5. Wall temperature gradient ($\theta'(0)$) versus Pr for varying $\lambda$ and $M$](image-url)
Figure 5 shows that \((-\dot{\theta}(0))\) increases with an increase in Prandtl number \(Pr\) and also with increase in permeability parameter \(\lambda\). On the other hand it decreases with an increase in Hartmann number \(M\).

Figure 6 clearly shows that the wall temperature gradient decreases with an increase in Radiation Parameter \(N\).

Figure 7 shows the variation of \((-\dot{\theta}(0))\) for varying values of Eckeret number \(Ec\). It is observed that the temperature gradient at the sheet decreases with an increase in Eckeret number \(Ec\).

Figure 8 shows the variation of skin friction with respect to the permeability parameter \(\lambda\) and indicates...
that it decreases with an increase in magnetic field parameter $M$.

5. **CONCLUSION**

In this study, forced convention is examined including the viscous heating and the magnetic field effects.

1. It is observed that the thickness of the thermal boundary layer, in the vicinity of the stretching sheet increases with an increase in Hartmann number $M$, Radiation parameter $N$ and Eckert number $Ec$.

2. It is also observed that the thickness of the thermal boundary layer decreases with an increase in permeability parameter $\lambda$ and Prandtl number $Pr$.

3. Rate of heat transfer at the wall increases with an increase in permeability parameter $\lambda$ and Prandtl number $Pr$.

4. Rate of heat transfer at the wall decreases with an increase in Hartmann number $M$, Radiation parameter $N$ and Eckert number $Ec$.

Skin friction coefficient decreases with an increase in magnetic field parameter $M$.

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