

Homotopy Perturbation Method for MHD Boundary Layer Flow with Low Pressure Gradient over a Flat Plate

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ABSTRACT

Using He's Homotopy Perturbation Method (HPM), the system of non-linear partial differential equations governing the MHD boundary layer equations with low pressure gradient over a flat plate are solved. The main advantage of HPM is that it does not require the small parameters in the equations and hence the limitations of traditional perturbation can be eliminated. The influence of various relevant physical characteristics are presented and discussed.

Keywords: MHD, Boundary layer flow, Homotopy perturbation method (HPM), Pressure gradient.

NOMENCLATURE

		p	pressure
B_0	constant applied magnetic field	Greek symbols	
E	electric field	η	dimensionless similarity variable
f	dimensionless stream function	ρ	density of fluid
Pr	prandtl number	ν	kinematic viscosity
u	velocity component in x direction	Ψ	stream function
U_∞	free stream velocity	α	thermal diffusivity
v	velocity component in y direction	θ	dimensionless temperature
x	coordinate in direction of surface motion	λ	pressure gradient parameter
y	coordinate in direction normal to surface ce motion	σ	electrical conductivity
M	dimensionless magnetic field parameter	Superscript	
T	absolute temperature	'	derivative with respect to η

Subscripts

- w properties at the plate
- ∞ free stream condition

1. INTRODUCTION

Study of magnetohydrodynamics (MHD) flow with heat and mass transfer play an important role in various industrial applications. Some important applications are cooling of nuclear reactors, liquid metals fluid, power generation system and aero dynamics. It is of importance in connection with many engineering problem, such as sustained plasma confinement for controlled thermo nuclear fusion and electromagnetic casting of metals. MHD finds applications in electromagnetic pumps, crystal growing, MHD coules and bearings, plasma jets and chemical synthesis.

Most of the problem in MHD area are nonlinear. Except a limited number of these problem have precise analytical solution, most of them do not have exact solution, and so these nonlinear equations should be solved using other proper methods. Most scientists believe that the combination of numerical and semi-exact analytical methods can lead to applicable results. In this paper one of the semi-exact method which is called HPM has been introduced and applied in MHD boundary layer flow with low pressure gradient over a flat plat. The initial work in HPM was studied by J. H. He (1998, 2000, 2001, 2005, 2009) and after that these investigations inspired a lot of researchers Ariel *et al.* (2006), Beléndez *et al.* (2008), Bird *et al.* (2007), Ganji and Ganji (2008), Ganji and Rajabi (2006), Ganji and Sadighi (2006), Hosein *et al.* (2008), Ma *et al.* (2008), Siddiqui *et al.* (2008), Zhang *et al.* (2008) and Zhang and He (2006) to solve nonlinear equations with this method.

The present paper deals with the steady MHD laminar forced convection with pith homotopy perturbation method (HPM). The influence of various relevant physical characteristics are presented are discussed.

Basic idea of homotopy perturbation method (HPM): To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation.

$$A(u) - f(r) = 0, r \in \Omega \tag{1}$$

Subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, r \in \Gamma \tag{2}$$

Where A is a general differential operator, B is a boundary operator, f(r) is a known analytical function and Γ is the boundary of the domain Ω . A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Therefore Eq.1 can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, r \in \Omega \tag{3}$$

By the homotopy perturbation technique, we construct a homotopy

$v(r, p): \Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0, 1], r \in \Omega \tag{4}$$

Where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation that satisfies the boundary condition. Obviously, from these definitions we will have:

$$H(v, 0) = L(v) - L(u_0) = 0$$

$$H(v, 1) = A(v) - f(r) = 0$$

The changing process of p from zero to one is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a “small parameter” and assuming that the solution of (4) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 \dots \tag{5}$$

Setting $p = 1$, results in the approximate solution of (1):

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \tag{6}$$

The convergence and stability of this method was shown by J. H. He (2005).

2. MATHEMATICAL FORMULATION

Consider steady flow, with constant free stream velocity U_∞ without turbulence over a semi infinite flat plate aligned with the flow. All fluid properties are considered to the constant, in the presence of a transverse magnetic field of uniform strength B_0 fixed to the plate.

We consider the case of a short circuit problem in which the applied electric field $E = 0$, and also assure that the induced magnetic field is small compared to the external magnetic field B_0 . This

implies a small magnetic Reynolds number, the Joule heating in the energy equation is neglected. The governing boundary layer equations are [Bird *et al.* (2007)]:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{7}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} - \frac{\sigma B_0^2}{\rho} u \tag{8}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{9}$$

Along with the boundary conditions for the problem are given by:

$$u(x, 0) = 0, v(x, 0) = 0, T(x, 0) = 0, u(x, \infty) \rightarrow U_\infty, T(x, \infty) \rightarrow T_\infty \tag{10}$$

The continuity Eq.7 is satisfied by introducing a stream function Ψ such that $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$

The momentum and energy equations can be transformed into the corresponding ordinary nonlinear differential equations by the following

transformation: $\eta = y \sqrt{\frac{U_\infty}{\nu x}}, \Psi = \sqrt{\nu x U_\infty} f(\eta),$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{11}$$

Where η is the independent similarity variable. The transformed nonlinear ordinary equations are

$$f''' + \frac{1}{2} f f'' - M^2 f' = -\lambda \tag{12}$$

$$\theta'' + \frac{Pr}{2} f f' = 0 \tag{13}$$

The transformed boundary conditions are:

$$f(0) = 0, f'(0) = 0, \theta(0) = 1, f'(\infty) \rightarrow 1, \theta(\infty) \rightarrow 0. \tag{14}$$

Where prime denotes differentiation with respect to η , $\lambda = -\frac{x}{\rho U_\infty} \frac{dp}{dx}$ is the pressure gradient, $M^2 = \frac{\sigma B_0^2 x}{\rho U_\infty}$ is the dimensionless magnetic parameter and $Pr = \nu/\alpha$ is the Prandtl number.

2.1. Solution with Homotopy Perturbation Method

According to the HPM, the homotopy form of Eq. (12) and (13) are constructed as follows:

$$(1-p)(f''' - M^2 f') + p(f''' + \frac{1}{2} f f'' - M^2 f') = -\lambda \tag{15}$$

$$(1-p)(\theta'') + p(\theta'' + \frac{Pr}{2} f f') = 0 \tag{16}$$

We consider f and θ as the following:

$$f = f_0 + p f_1 + p^2 f_2 \dots \tag{17}$$

$$\theta = \theta_0 + p \theta_1 + p^2 \theta_2 \dots$$

By substituting Eq. (17) into (15) and (16), and then (I) Terms independent of p give

$$f_0''' - M^2 f_0' = -\lambda \tag{18}$$

$$\theta_0'' = 0 \tag{19}$$

The boundary conditions are

$$f_0(0) = 0, f_0'(0) = 0, f_0'(\infty) = 1, \theta_0(0) = 1, \theta_0(\infty) = 0. \tag{20}$$

(II) Terms containing only p give

$$f_1''' - M^2 f_1' + \frac{1}{2} f_0 f_0'' = 0 \tag{21}$$

$$\theta_1'' + \frac{Pr}{2} f_0 \theta_0 = 0 \tag{22}$$

The boundary conditions are

$$f_1(0) = 0, f_1'(0) = 0, f_1'(\infty) = 0, \theta_1(0) = 0, \theta_1(\infty) = 0 \tag{23}$$

(III) Terms containing only p^2 give

$$f_2''' - M^2 f_2' + \frac{1}{2} (f_0 f_1'' + f_1 f_0'') = 0 \tag{24}$$

$$\theta_2'' + \frac{Pr}{2} (f_0 \theta_1' + f_1 \theta_0') = 0 \tag{25}$$

The boundary conditions are

$$f_2(0) = 0, f_2'(0) = 0, f_2'(\infty) = 0, \theta_2(0) = 0, \theta_2(\infty) = 0 \tag{26}$$

Solving Eq. (18)-(19), (21)-(22) and (24)-(25) with boundary conditions (20), (23) and (26) respectively, we have

$$f_0 = C_1 + C_2 e^{m\eta} + C_3 e^{-m\eta} + \frac{\lambda \eta}{M^2} \tag{27}$$

$$f_1 = C_4 + C_5 e^{m\eta} + C_6 e^{-m\eta} + A_3 \eta e^{m\eta} + A_4 \eta e^{-m\eta} + A_5 \eta^2 e^{m\eta} + A_6 \eta^2 e^{-m\eta} + A_7 e^{2m\eta} + A_8 e^{-2m\eta} + A_9 \eta \tag{28}$$

$$f_2 = C_7 + C_8 e^{m\eta} + C_9 e^{-m\eta} + A_{11} \eta e^{m\eta} + A_{12} \eta e^{-m\eta} + A_{13} \eta^2 e^{m\eta} + A_{14} \eta^2 e^{-m\eta} + A_{15} \eta^3 e^{m\eta} + A_{16} \eta^3 e^{-m\eta} + A_{17} \eta^4 e^{m\eta} + A_{18} \eta^4 e^{-m\eta} + A_{19} e^{2m\eta} + A_{20} e^{-2m\eta} + A_{21} \eta e^{2m\eta} + A_{22} \eta e^{-2m\eta} + A_{23} \eta^2 e^{2m\eta} + A_{24} \eta^2 e^{-2m\eta} + A_{25} e^{3m\eta} + A_{26} e^{-3m\eta} + A_{27} \eta^3 + A_{28} \eta^2 + A_{29} \eta \tag{29}$$

$$\theta_0 = -\frac{\eta}{6} + 1 \tag{30}$$

$$\theta_1 = -\frac{Pr}{2M^2} \left(\frac{1}{3M} + 1 \right) C_2 e^{M\eta} + \frac{Pr}{2M^2} \left(\frac{1}{3M} - 1 \right) C_3 e^{-M\eta} + \frac{C_2 Pr}{12M^2} \eta e^{M\eta} + \frac{C_3 Pr}{2M^2} \eta e^{-M\eta} + \frac{\lambda Pr}{144M^2} \eta^4 - \frac{Pr}{12} \left(\frac{\lambda}{M^2} - \frac{C_1}{6} \right) \eta^3 - \frac{C_1 Pr}{4} \eta^2 + B_1 \eta + B_2 \tag{31}$$

$$\theta_2 = \left(\frac{B_3}{M^2} - \frac{2B_5}{M^3} + \frac{6B_7}{M^4} - \frac{24B_9}{M^5} \right) e^{M\eta} + \left(\frac{B_4}{M^2} - \frac{2B_6}{M^3} + \frac{6B_8}{M^4} + \frac{24B_{10}}{M^5} \right) e^{-M\eta} + \left(\frac{B_5}{M^2} - \frac{4B_7}{M^3} + \frac{18B_9}{M^4} \right) \eta e^{M\eta} + \left(\frac{B_6}{M^2} + \frac{4B_8}{M^3} + \frac{18B_{10}}{M^4} \right) \eta e^{-M\eta} + \left(\frac{B_7}{M^2} - \frac{6B_9}{M^3} \right) \eta^2 e^{M\eta} + \left(\frac{B_8}{M^2} + \frac{6B_{10}}{M^3} \right) \eta^2 e^{-M\eta} + \frac{B_9}{M^2} \eta^3 e^{M\eta} + \frac{B_{10}}{M^2} \eta^3 e^{-M\eta} + \left(\frac{B_{11}}{4M^2} - \frac{B_{13}}{M^3} \right) e^{2M\eta} + \left(\frac{B_{12}}{4M^2} + \frac{B_{14}}{4M^3} \right) e^{-2M\eta} +$$

$$\frac{B_{13}}{4M^2}\eta e^{2M\eta} + \frac{B_{14}}{4M^2}\eta e^{-2M\eta} + \frac{B_{15}}{30}\eta^6 + \frac{B_{16}}{20}\eta^5 + \frac{B_{17}}{12}\eta^4 + \frac{B_{18}}{6}\eta^3 + \frac{B_{19}}{2}\eta^2 + B_{20}\eta + B_{21} \quad \dots(32)$$

The constant coefficients, can be calculated using boundary conditions, the boundary condition $\eta=\infty$ were replaced by those at $\eta=6$ in accordance with standard practice in the boundary layer analysis. If $p \rightarrow 1$, we find the approximate solution of Eq. (15) and (16).

The constant coefficients $A_i (i = 1, 2, 3, \dots, 29), B_j (j = 1, 2, 3, \dots, 21)$ and $C_k (k = 1, 2, 3, 4, 5, 6)$ are defined as in the Appendix.

3. RESULTS AND DISCUSSION

The numerical results are obtained for low pressure gradient (λ) 0.05 and 0.1 and fixed value of Prandtl number (Pr) 1.0 and for various values of magnetic parameter M. The effect of magnetic parameter M on the velocity and temperature are presented in Fig. 1 to 4.

On the other hand, the skin friction coefficient $f''(0)$ and heat transfer coefficient $-\theta'(0)$ against magnetic parameter (M), for Pr=1.0 and $\lambda=0.05$ and 0.1 are presented in Fig. 5 and 6 respectively.

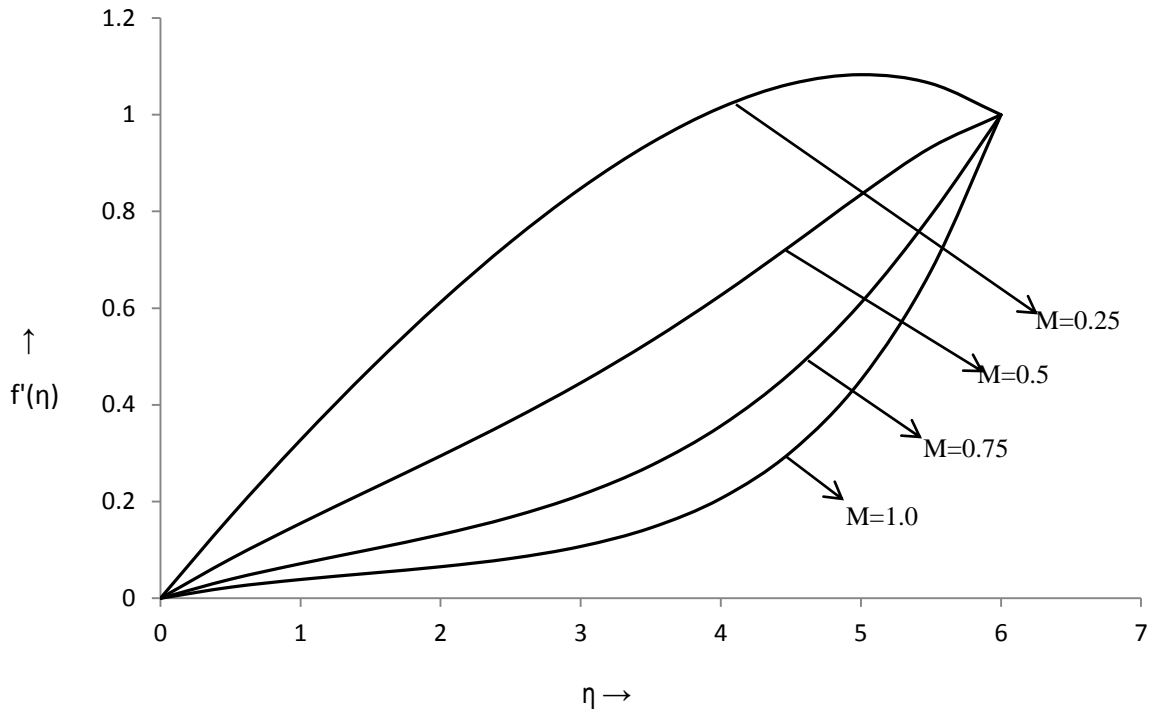


Fig. 1. Velocity profiles f' for various values of magnetic parameter M when Pr=1.0 and $\lambda=0.05$.

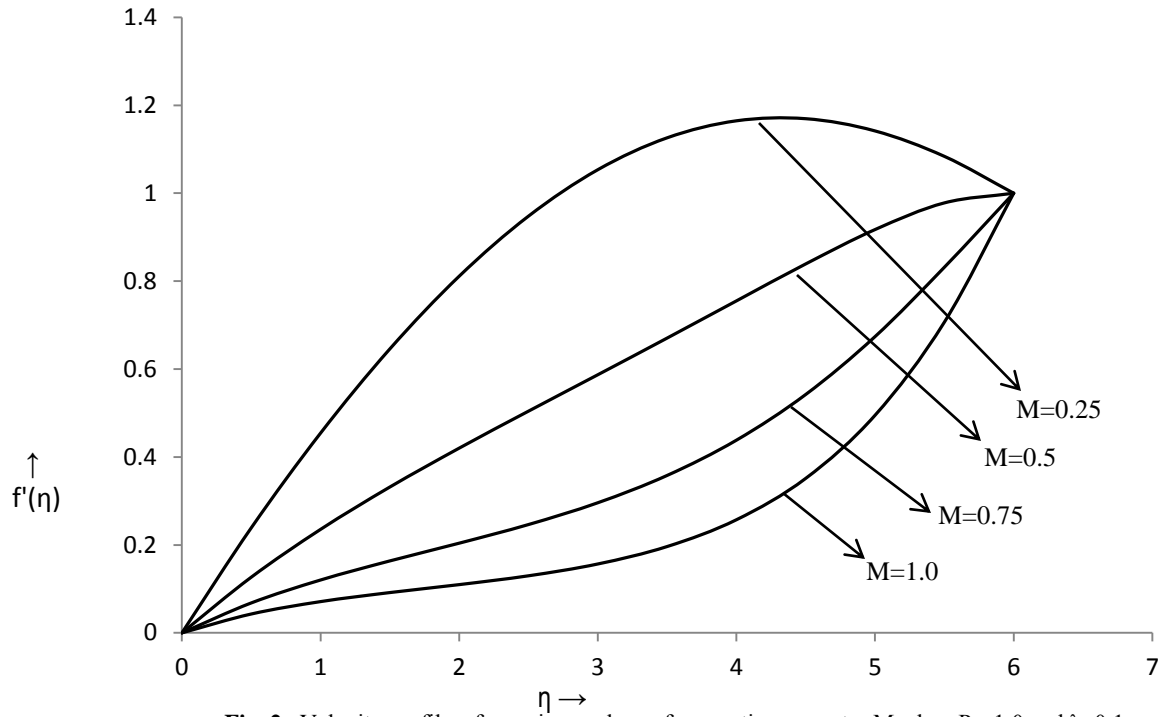


Fig. 2 . Velocity profiles for various values of magnetic parameter M when $Pr=1.0$ and $\lambda=0.1$.

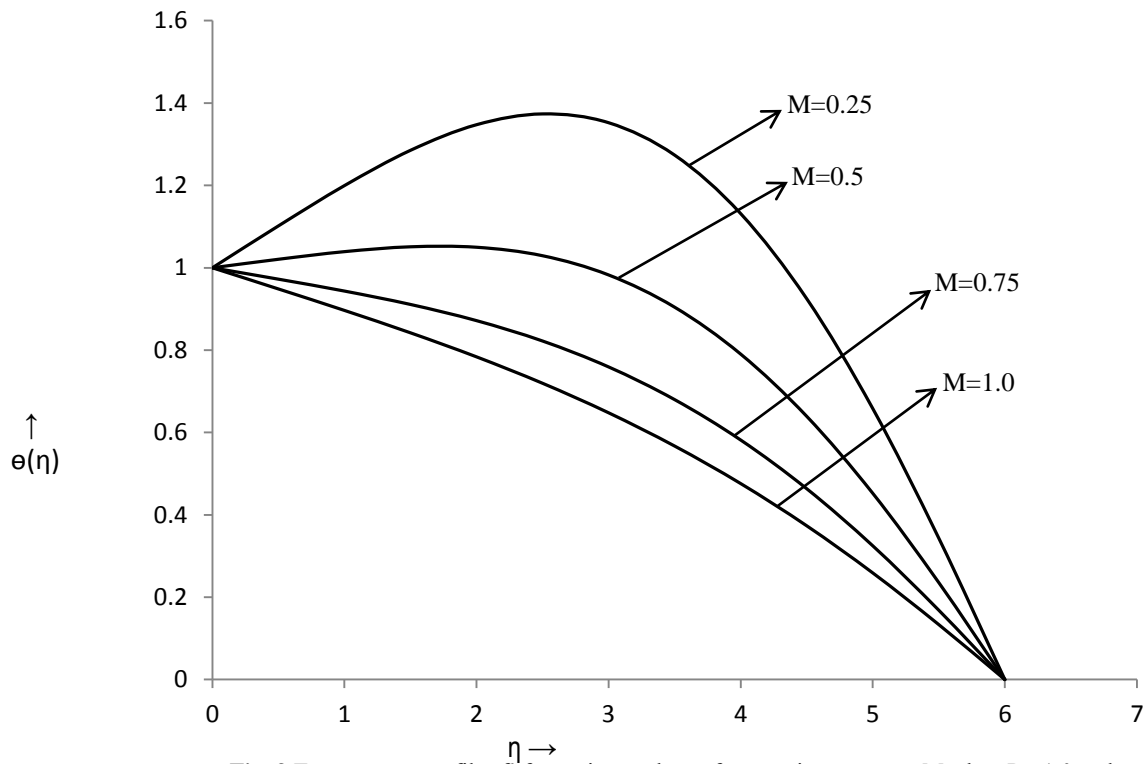


Fig. 3. Temperature profiles θ for various values of magnetic parameter M when $Pr=1.0$ and $\lambda=0.05$

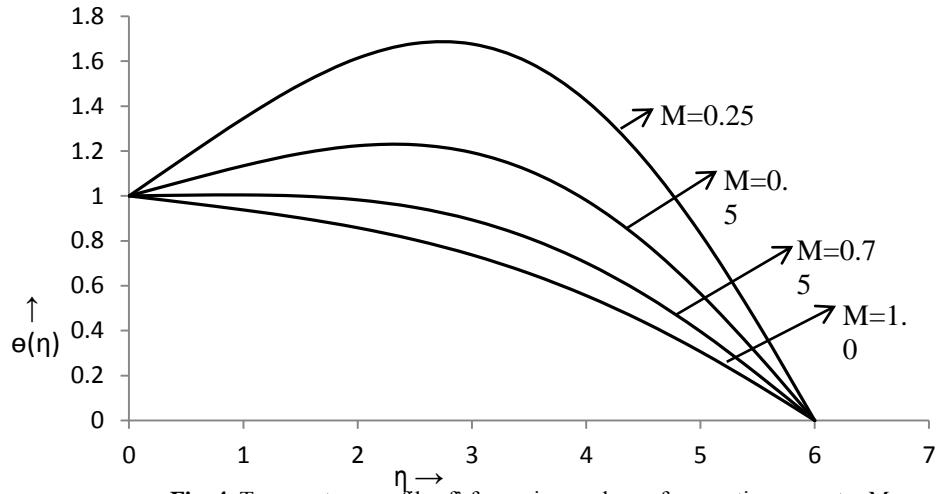


Fig. 4. Temperature profiles θ' for various values of magnetic parameter M when $Pr=1.0$ and $\lambda=0.1$

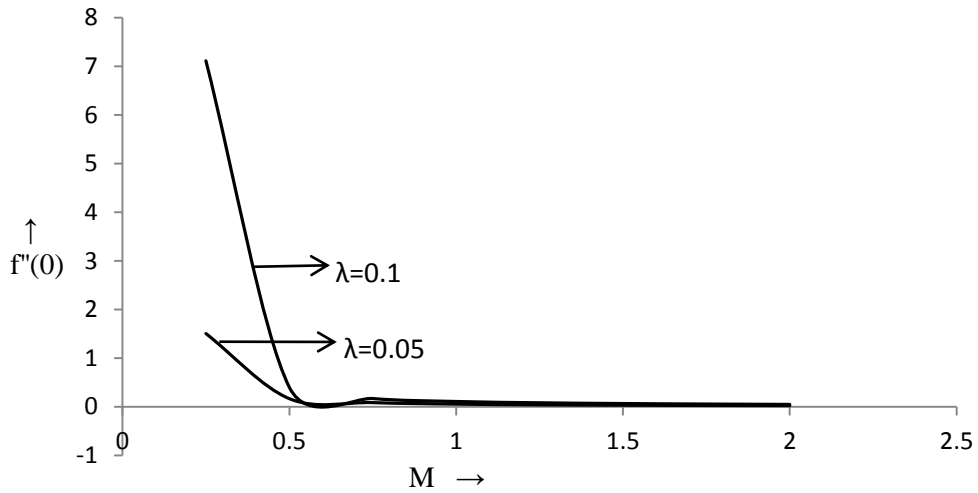


Fig. 5. Coefficient of skin friction against magnetic parameter for $Pr=1.0$

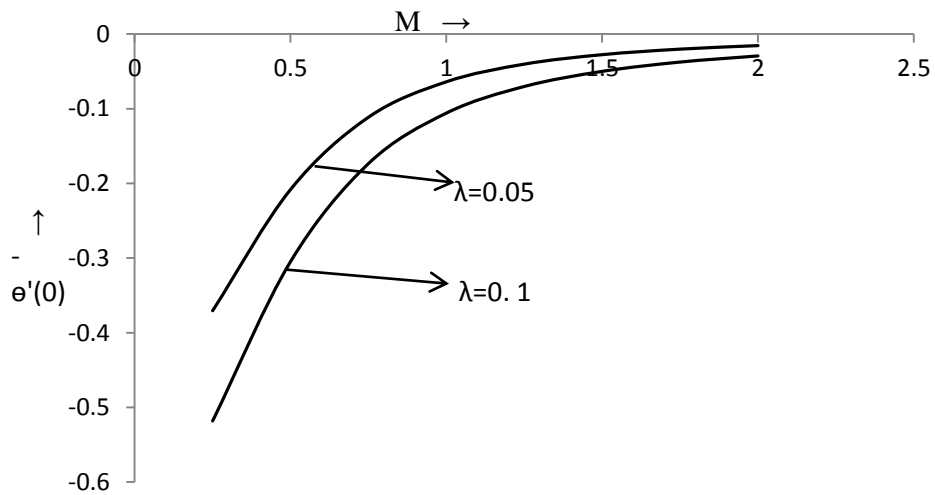


Fig. 6. Nusselt number against magnetic parameter for $Pr=1.0$

4. CONCLUSION

In this study, the momentum and energy equations, was solved with homotopy perturbation method (HPM) for MHD boundary layer flow with low pressure gradient over a flat plate.

We notice from figures 1 to 4 that for given λ the velocity and thermal boundary layer thicknesses decrease with the increasing M.

The skin friction and Nusselt number against the magnetic interaction parameter M, for Pr=1.0 are shown in figures 5 and 6 respectively. It is noted that for increasing value of λ , the skin friction increases but it decreases with the increasing values of M, whereas opposite phenomenon occurs in Nusselt number.

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APPENDIX

$$A_1 = e^{M\eta} - e^{-M\eta},$$

$$A_2 = \frac{\lambda}{M^2} (1 - e^{-M\eta}),$$

$$C_2 = \frac{1-A_2}{MA_1},$$

$$C_3 = \frac{1-A_2}{MA_1} + \frac{\lambda}{M^3},$$

$$C_1 = -(C_2 + C_3),$$

$$A_3 = -\frac{C_1 C_2}{4} - \frac{3\lambda C_2}{4M^5},$$

$$A_4 = -\frac{C_1 C_3}{4} + \frac{3\lambda C_3}{4M^5},$$

$$A_5 = \frac{\lambda C_2}{4M^4},$$

$$A_6 = \frac{\lambda C_3}{4M^4},$$

$$A_7 = -\frac{C_2^2}{12M},$$

$$A_8 = \frac{C_3^2}{12M},$$

$$A_9 = -\frac{2C_2 C_3}{M^2},$$

$$A_{10} = (A_3 + 6MA_3 + 12A_5 + 36MA_6)e^{6M} + (A_4 - 6MA_4 + 12A_6 - 36MA_6)e^{-6M} + 2MA_7e^{12M} - 2MA_8e^{-12M} + A_9$$

$$C_5 = \frac{(A_3 + A_4 + 2MA_7 - 2MA_6 + A_9)e^{-6M}}{MA_1} - \frac{A_{10}}{MA_1},$$

$$C_6 = (C_5 + 2A_7 - 2A_8) + \frac{1}{M}(A_3 + A_4 + A_9),$$

$$C_4 = -(C_5 + C_6 + A_7 + A_8),$$

$$A_{11} = -\frac{1}{4M^2} \left\{ C_1(M^2C_5 + 2MA_3 + 2A_5) + 5C_3M^2A_7 + M^2C_2C_4 \right. \\ \left. + \frac{3}{8M^3} \left\{ C_1(M^2A_3 + 4MA_5) + C_2M^2A_9 + \frac{\lambda}{M^2}(M^2C_5 + 2MA_3 + 2A_5) \right\} \right. \\ \left. - \frac{7}{8M^4} \left\{ C_1M^2A_5 + \frac{\lambda}{M^2}(M^2A_3 + 4MA_5) \right\} + \frac{45\lambda A_5}{16M^5} \right\}$$

$$A_{12} = -\frac{1}{4M^2} \left\{ C_1(-M^2C_6 - 2MA_4 + 2A_6) + 5C_2M^2A_8 + M^2C_3C_4 \right. \\ \left. - \frac{3}{8M^3} \left\{ C_1(M^2A_4 - 4MA_6) + C_3M^2A_9 + \frac{\lambda}{M^2}(-M^2C_6 - 2MA_4 + 2A_6) \right\} \right. \\ \left. - \frac{7}{8M^4} \left\{ C_1M^2A_6 + \frac{\lambda}{M^2}(M^2A_4 - 4MA_6) \right\} - \frac{45\lambda A_6}{16M^5} \right\}$$

$$A_{13} = -\frac{1}{8M^2} \left\{ C_1(M^2A_3 + 4MA_5) + C_2M^2A_9 + \frac{\lambda}{M^2}(M^2C_5 + 2MA_3 + 2A_5) \right\} \\ + \frac{3}{8M^3} \left\{ C_1M^2A_5 + \frac{\lambda}{M^2}(M^2A_3 + 4MA_5) \right\} - \frac{21\lambda A_5}{16M^4}$$

$$A_{14} = -\frac{1}{8M^2} \left\{ C_1(M^2A_4 - 4MA_6) + C_3M^2A_9 + \frac{\lambda}{M^2}(-M^2C_6 - 2MA_4 + 2A_6) \right\} - \frac{3}{8M^3} \left\{ C_1M^2A_6 + \frac{\lambda}{M^2}(M^2A_4 - 4MA_6) \right\} - \frac{21\lambda A_6}{16M^4}$$

$$A_{15} = -\frac{1}{12M^2} \left\{ M^2A_5C_1 + \frac{\lambda}{M^2}(M^2A_3 + 4MA_5) \right\} + \frac{3\lambda A_5}{8M^3}$$

$$A_{16} = -\frac{1}{12M^2} \left\{ M^2A_6C_1 + \frac{\lambda}{M^2}(M^2A_4 - 4MA_6) \right\} - \frac{3\lambda A_6}{8M^3}$$

$$A_{17} = -\frac{1\lambda A_5}{16M^2},$$

$$A_{18} = -\frac{1\lambda A_6}{16M^2}$$

$$A_{19} = -\frac{1}{12M^3} \left\{ C_2C_5M^2 + 4C_1A_7M^2 + C_2(M^2C_5 + 2MA_3 + 2A_5) \right\} + \frac{11}{72M^4} \left\{ C_2M^2A_3 + C_2(M^2A_3 + 4MA_5) + 4\lambda A_7 \right\} - \frac{85C_2A_5}{108M^3}$$

$$A_{20} = \frac{1}{12M^3} \left\{ C_3C_6M^2 + 4C_1A_8M^2 + C_3(-M^2C_6 - 2MA_4 + 2A_6) \right\} + \frac{11}{72M^4} \left\{ C_3M^2A_4 + C_3(M^2A_4 - 4MA_6) + 4\lambda A_8 \right\} + \frac{85C_3A_6}{108M^3}$$

$$A_{21} = -\frac{1}{12M^3} \left\{ C_2A_3M^2 + 4\lambda A_7 + C_2(M^2A_3 + 4MA_5) \right\} + \frac{11C_2A_5}{18M^2},$$

$$A_{22} = \frac{1}{12M^3} \left\{ C_3A_4M^2 + 4\lambda A_8 + C_3(M^2A_4 - 4MA_6) \right\} + \frac{11C_3A_6}{18M^2},$$

$$A_{23} = -\frac{C_2A_5}{6M},$$

$$A_{24} = \frac{C_3A_6}{6M},$$

$$A_{25} = -\frac{5C_2A_7}{48M},$$

$$A_{26} = \frac{5C_3A_8}{48M},$$

$$A_{27} = \frac{1}{3}(C_2A_6 + C_3A_5),$$

$$A_{28} = \frac{1}{2M^2} \left\{ C_3(M^2A_3 + 2MA_5) + C_2(M^2A_4 - 2MA_6) \right\},$$

$$A_{29} = \frac{2}{M^2}(C_2A_6 + C_3A_5) + \frac{1}{M^2} \left\{ C_2(-M^2C_6 - 2MA_4 + 2A_6) + C_3(M^2C_5 + 2MA_3 + 2A_5) + C_2C_6M^2 + C_3C_5M^2 \right\},$$

$$B_1 = \frac{C_2Pr}{36M^3} e^{6M} - \frac{C_3Pr}{36M^3} e^{-6M} + \frac{3\lambda Pr}{2M^2} + C_1Pr - \frac{B_2}{6},$$

$$B_2 = \frac{C_2Pr}{2M^2} \left(\frac{1}{3M} + 1 \right) - \frac{C_3Pr}{2M^2} \left(\frac{1}{3M} - 1 \right),$$

$$B_3 = -\frac{Pr}{2} \left\{ -\frac{C_1C_2Pr}{2M} \left(\frac{1}{3M} + 1 \right) - \frac{C_1C_2Pr}{12M^4} + C_2B_1 - \frac{C_5}{6} \right\},$$

$$B_4 = -\frac{Pr}{2} \left\{ -\frac{C_1C_3Pr}{2M} \left(\frac{1}{3M} - 1 \right) - \frac{C_1C_3Pr}{12M^4} + C_3B_1 - \frac{C_6}{6} \right\},$$

$$B_5 = -\frac{Pr}{2} \left\{ -\frac{C_2\lambda Pr}{2M^3} \left(\frac{1}{3M} + 1 \right) + \frac{C_1C_2Pr}{12M^3} - \frac{C_1C_2Pr}{2} - \frac{C_3\lambda Pr}{12M^6} - \frac{A_3}{6} \right\},$$

$$B_6 = -\frac{Pr}{2} \left\{ -\frac{C_1C_3Pr}{12M^3} - \frac{C_1C_3Pr}{2} - \frac{C_3\lambda Pr}{2M^3} \left(\frac{1}{3M} - 1 \right) - \frac{C_3\lambda Pr}{12M^6} - \frac{A_4}{6} \right\},$$

$$B_7 = -\frac{Pr}{2} \left\{ -\frac{C_2Pr}{4} \left(\frac{\lambda}{M^2} - \frac{C_1}{6} \right) + \frac{C_2\lambda Pr}{12M^5} - \frac{A_5}{6} \right\},$$

$$B_8 = -\frac{Pr}{2} \left\{ -\frac{C_3Pr}{4} \left(\frac{\lambda}{M^2} - \frac{C_1}{6} \right) - \frac{C_3\lambda Pr}{12M^5} - \frac{A_6}{6} \right\},$$

$$B_9 = -\frac{Pr}{2} \frac{C_2\lambda Pr}{36M^2},$$

$$B_{10} = -\frac{Pr}{2} \frac{C_3\lambda Pr}{36M^2},$$

$$B_{11} = -\frac{Pr}{2} \left\{ -\frac{C_2^2Pr}{2M} \left(\frac{1}{3M} + 1 \right) - \frac{C_2^2Pr}{12M^4} - \frac{A_7}{6} \right\},$$

$$B_{12} = -\frac{Pr}{2} \left\{ -\frac{C_3^2Pr}{2M} \left(\frac{1}{3M} - 1 \right) - \frac{C_3^2Pr}{12M^4} - \frac{A_8}{6} \right\},$$

$$B_{13} = -\frac{Pr}{2} \frac{C_2^2Pr}{12M^3},$$

$$B_{14} = \frac{Pr}{2} \frac{C_3^2Pr}{12M^3},$$

$$B_{15} = \frac{Pr}{2} \frac{\lambda^2 Pr}{36M^4},$$

$$B_{16} = -\frac{Pr}{2} \left\{ \frac{C_1\lambda Pr}{36M^2} - \frac{\lambda Pr}{4M^2} \left(\frac{\lambda}{M^2} - \frac{C_1}{6} \right) \right\},$$

$$B_{17} = -\frac{Pr}{2} \left\{ -\frac{C_1\lambda Pr}{2M^2} - \frac{C_1Pr}{4} \left(\frac{\lambda}{M^2} - \frac{C_1}{6} \right) \right\},$$

$$\begin{aligned}
 B_{18} &= -\frac{Pr}{2} \left\{ -\frac{C_1^2 Pr}{2} + \frac{\lambda B_1}{M^2} - \frac{A_9}{6} \right\}, \\
 B_{19} &= -\frac{Pr}{2} \left\{ -\frac{C_2 C_3 Pr}{2M} \left(\frac{1}{3M} - 1 \right) - \frac{C_2 C_3 Pr}{6M^4} + C_1 B_1 - \right. \\
 &\quad \left. \frac{C_2 C_3 Pr}{2M} \left(\frac{1}{3M} + 1 \right) - \frac{C_4}{6} \right\}, \\
 B_{20} &= \frac{1}{6} \left(-\frac{B_3}{M^2} + \frac{2B_5}{M^3} - \frac{6B_7}{M^4} + \frac{24B_9}{M^5} - \frac{6B_5}{M^2} + \frac{24B_7}{M^3} - \right. \\
 &\quad \left. \frac{108B_9}{M^4} - \frac{36B_7}{M^2} - \frac{216B_9}{M^3} - \frac{216B_9}{M^2} \right) e^{6M} + \frac{1}{6} \left(-\frac{B_4}{M^2} + \frac{2B_6}{M^3} - \right. \\
 &\quad \left. \frac{6B_8}{M^4} - \frac{24B_{10}}{M^5} - \frac{6B_5}{M^2} - \frac{24B_8}{M^3} - \frac{108B_{10}}{M^4} - \frac{36B_8}{M^2} - \frac{216B_{10}}{M^3} - \right. \\
 &\quad \left. \frac{216B_{10}}{M^2} \right) e^{-6M} + \frac{1}{6} \left(-\frac{B_{11}}{4M^2} + \frac{B_{13}}{4M^3} - \frac{3B_{13}}{2M^2} \right) e^{12M} + \\
 &\quad \left(-\frac{B_{12}}{4M^2} - \frac{B_{14}}{4M^3} - \frac{3B_{14}}{2M^2} \right) e^{-12M} - \frac{7776}{5} B_{15} - \\
 &\quad \frac{1944}{5} B_{16} - 108B_{17} - 136B_{18} - -18B_{19} - B_{21}, \\
 B_{21} &= -\frac{1}{M^2} \left(B_3 + B_4 + \frac{B_{11}}{4} + \frac{B_{12}}{4} \right) \\
 &\quad + \frac{1}{M^3} \left(2B_5 + 2B_6 + \frac{B_{13}}{4} - \frac{B_{14}}{4} \right) \\
 &\quad - \frac{6}{M^4} (B_7 + B_8) + \frac{24}{M^5} (B_9 - B_{10})
 \end{aligned}$$