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The Influence of Ekman Number on Flows Over an Oscillating Isothermal Vertical Plate in a Rotating Frame

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ABSTRACT

Unsteady flows of a Newtonian fluid past an oscillating infinite isothermal vertical plate in a rotating frame are studied. Two closed forms of the velocity field are determined by means of Laplace transform method and by the technique of coupling of the homotopy perturbation method with the Laplace transform method. The equivalence of the obtained expressions for the velocity is provided. The friction coefficients are determined and, the influence of Ekman number on the velocity field is analyzed by graphical illustrations. It is obtained that, the faster frame rotations reduce the thickness of moving fluid layer.

Keywords: Isothermal plate; Oscillating flows; Rotating fluid; Ekman number.

NOMENCLATURE

C_p	specific heat at constant pressure
Ek	Ekman number
g	gravitational acceleration
Gr	Grashof number
Im	Imaginary part of a complex number
J_n	Bessel function
k	thermal conductivity
Pr	Prandtl number
Re	real part of a complex number
s	Laplace transform parameter
T	fluid temperature
T_w	wall temperature
T_∞	temperature far away from the plate
u, v	velocity components along x and ydirection
μ	dynamic viscosity
ν	kinematic viscosity
Ω	angular velocity of the frame
ω	oscillation frequency
θ	non-dimensional temperature
τ	shear stress

1. INTRODUCTION

Rotating flow is an important branch of fluid dynamics. In many practical applications, thermal rotating flows occur in a variety of rotating machinery. Also, some natural phenomena such as geophysical systems, tornadoes, hurricanes, ocean circulations imply rotating flows with heat and mass transfer. In a non-isothermal flow field, the fluid density varies spatially with the local temperature. If the temperature gradient is normal to the body forces in the field, the buoyancy - driven fluid motion emerges. Several books and articles on hydrodynamic and heat transfer characteristics of rotating flows have appeared (Greenspan (1968), Owen and Rogers (1989), Soong and Ma (1995), Soong (2001), Muthucumaraswamy *et al* (2013)). Rotating free - disk flow was first analyzed by von Karman (1921) and, the class of thermally induced flows related to a single disk rotating with its environment was studied by Riley (1967) using compressible boundary layer flow equations. The same problem was studied by Hudson (1968) by invoking the Boussinesq approximation. The flow and heat transfer problem due to moving surfaces has many practical applications, such as in polymer processing systems, production of paper, insulating material, etc. Some interesting problems were studied in the references (Liao (2003), Kumari and Nath (2005), Magyari and Keller (2005), Siddheshwar *et al* (2014)). Given their importance for theoretical and practical problems, the flows in rotating frame with various mechanical and thermal conditions were addressed by many researchers (Puri (1974), Hayat *et al* (2001), Abelman *et al* (2009), Salah *et al* (2011, a, b, c), Salah *et al* (2013)). The study of fluid flow past a heated surface has attracted the interest of many investigators in view of its important applications in many engineering problems such as petroleum industries, cooling of nuclear reactors, the boundary layer control in aerodynamics, etc. Until recently, this study has been largely concerned with flow and heat transfer characteristics in various physical aspects (Chamkha *et al* (2001), Abd-El-Naby *et al* (2003), Chen (2004), Vasu *et al* (2011), Narahary et al (2011), Fetecau et al (2013), Samiulhaq *et al* (2014)). In this paper the flow of an incompressible, homogeneous, Newtonian fluid near an infinite isothermal vertical plate is studied. The fluid and plate are in the state of rigid body rotation around z - axis, perpendicular to the plate. In the same time, the plate oscillates in its plane along the x-direction (Fig.1). The basic equations corresponding to the considered problem, with appropriate initial-boundary conditions, are solved by means of two methods, namely, the Laplace transform method respectively, by technique of coupling of the previous method with the homotopy perturbation method. The equivalence of the two forms of the obtained solutions is demonstrated and, the friction coefficients are determined. Finally, the influence of the Ekman number on the fluid motion is analyzed by graphical illustrations.

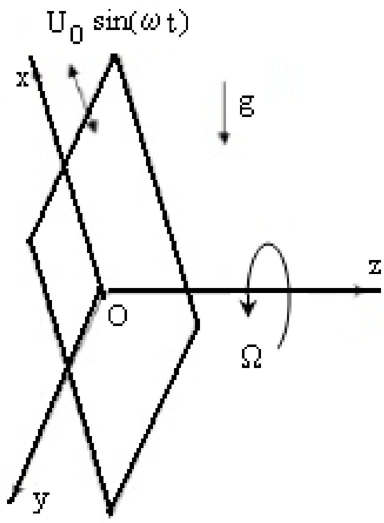

Fig. 1. Geometry flow.

2. FORMULATION OF THE PROBLEM AND SOLUTION

We consider unsteady flows of an incompressible, homogeneous Newtonian fluid, induced in a semi-infinite expanse bounded by an infinite plate situated in the plane $z = 0$ of a Cartesian coordinate system *Oxyz*. Initially, at $t = 0$, the plate and the fluid are at rest with the same temperature, $T\infty$ .After this moment, the fluid and plate are in the state of rigid body rotation with a constant angular velocity $\vec{\Omega} = \Omega\vec{k}$, $\vec{k}$ being a unit vector parallel with z-axis.

The fluid occupies the half-space $z \geq 0$, the plate oscillates in its plane along the x- axis and non-slip condition on the plate is accepted. Since, the plate is represented by the (x, y) - plane, all physical variables are functions of z and t only.
The equations governing the considered flow are given by (Hayat *et al* (2001), Chamkha *et al* (2001)).

$$\frac{\partial u}{\partial t} - 2\Omega v = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial z^2}, \quad (1)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = \nu\frac{\partial^2 v}{\partial z^2}, \quad (2)$$

$$\rho C_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2}, \quad (3)$$

where ($u(z, t)$, $v(z, t)$) are the velocity components along the x-axis and y-axis respectively, g the gravitational acceleration, β coefficient of volume expansion, ν kinematic viscosity, ρ the density, Cp the specific heat at constant pressure, κ the coefficient of thermal conductivity and $T(z, t)$ is the temperature of the fluid. Initial and boundary conditions are:

$$u = 0, v = 0, T = T_\infty, t \leq 0, \quad (4)$$

$$u = U_0 \sin(\omega t), v = 0, T = T_w > T_\infty, z = 0, t > 0, \quad (5)$$

$$u = 0, v = 0, T = T_\infty\ z \to \infty, t \geq 0. \quad (6)$$

We use the following set of non-dimensional variables and functions:

$$u' = \frac{u}{U_0}, v' = \frac{v}{U_0}, t' = \frac{tGrU_0^2}{\nu}, z' = \frac{z\sqrt{Gr}U_0}{\nu},$$

$$\omega' = \frac{\nu\omega}{GrU_0^2}, Pr = \frac{\mu C_p}{k}, Gr = \frac{\nu g\beta(T_w - T_\infty)}{U_0^3},$$

$$Ek = \frac{\nu\Omega}{GrU_0^2}, \theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad (7)$$

where *Pr* is the Prandtl number, *Gr is* the Grashof number and *Ek* is the Ekman number. Dropping prime notations, the set of non-dimensional partial differential equations is

$$\frac{\partial u}{\partial t} - 2Ekv = \theta + \frac{\partial^2 u}{\partial z^2}, \quad (8)$$

$$\frac{\partial v}{\partial t} + 2Eku = \frac{\partial^2 v}{\partial z^2}, \quad (9)$$

$$Pr\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2}, \quad (10)$$

with the initial and boundary conditions

$$u = 0, v = 0, \theta = 0, t \leq 0, \quad (11)$$

$$u = \sin(\omega t), v = 0, \theta = 1, z = 0, t > 0, \quad (12)$$

$$u = 0, v = 0, \theta = 0, z \to \infty, t \geq 0. \quad (13)$$

2.1 Laplace Transform Technique

In this section we determine solution of the problem (8)-(13) by using the Laplace transform method (Prudnikov *et al* (1992) a, b). Applying Laplace transform to the problem (10), $(11)_3$, $(12)_3$, $(13)_3$, we obtain the known expressions of the fluid temperature

$$\bar{\theta}(z,s)=\frac{1}{s}\exp(-z\sqrt{\Pr s}),\ \theta(z,t)=erfc\left(\frac{\sqrt{\Pr}z}{2\sqrt{t}}\right), \tag{14}$$

where $\bar{\theta}(z,s)$ is the Laplace transform of the function $\theta(z,\ t)$ and s is the transform parameter. Introducing the complex velocity field $q(z,\ t)=u(z,\ t)+iv(z,\ t)$ and then applying Laplace transform to the resulting equations, we obtain the transformed problems:

$$\frac{\partial q(z,t)}{\partial t}+2iEkq(z,t)=\theta(z,t)+\frac{\partial^2 q(z,t)}{\partial z^2}, \tag{15}$$

$$q(0,t)=\sin(\omega t),\ \lim_{z\to\infty}q(z,t)=0, \tag{16}$$

$$\frac{\partial^2\bar{q}(z,s)}{\partial z^2}-(s+2iEk)\bar{q}(z,s)=\frac{-1}{s}e^{-z\sqrt{Pr\,s}}, \tag{17}$$

$$\bar{q}(0,s)=\frac{\omega}{s^2+\omega^2},\ \lim_{z\to\infty}\bar{q}(z,s)=0. \tag{18}$$

The problem given by Eqs. (17) – (18) has the solution

$$\bar{q}(z,s)=\frac{\omega}{s^2+\omega^2}exp(-z\sqrt{s+2iEk})+\frac{exp(-z\sqrt{s+2iEk})-exp(-z\sqrt{Pr\,s})}{s[(Pr-1)s-2iEk]}, \tag{19}$$

which can be written in a suitable form as

$$\begin{aligned}\bar{q}(z,s)=&\frac{i\,exp(-z\sqrt{s+2iEk})}{2(s+\omega i)}-\frac{i\,exp(-z\sqrt{s+2iEk})}{2(s-\omega i)}\\&+\frac{i\,exp(-z\sqrt{s+2iEk})}{2Eks}\\&-\frac{(Pr-1)i\,exp(-z\sqrt{s+2iEk})}{2Ek[(Pr-1)s-2iEk]}\\&-\frac{i\,exp(-z\sqrt{Pr\,s})}{2Eks}+\frac{(Pr-1)i\,exp(-z\sqrt{Pr\,s})}{2Ek[(Pr-1)s-2iEk]}.\end{aligned} \tag{20}$$

By inverting the Laplace transform given by Eq. (20) (Prudnikov (1992) b, Hetnarski (1975)), we get the complex velocity field

$$\begin{aligned}q(z,t)=&\frac{i}{4}e^{-i\omega t}\varphi(z,t;1,\xi_1)-\frac{i}{4}e^{i\omega t}\varphi(z,t;1,\xi_2)\\&+\frac{i}{4Ek}\varphi(z,t;1,\sqrt{2iEk})\\&-\frac{i}{4Ek}e^{at}\varphi(z,t;1,\sqrt{a\Pr})\\&+\frac{i}{4Ek}e^{at}\varphi(z,t;\sqrt{\Pr},\sqrt{a})\\&-\frac{i}{2Ek}erfc\left(\frac{z\sqrt{\Pr}}{2\sqrt{t}}\right),\end{aligned} \tag{21}$$

where

$$\begin{aligned}\varphi(z,t;\alpha_1,\alpha_2)=&e^{-\alpha_1\alpha_2 z}erfc\left(\frac{\alpha_1 z}{2\sqrt{t}}-\alpha_2\sqrt{t}\right)\\&+e^{\alpha_1\alpha_2 z}erfc\left(\frac{\alpha_1 z}{2\sqrt{t}}+\alpha_2\sqrt{t}\right),\end{aligned} \tag{22}$$

$$\xi_1=\sqrt{(2Ek-\omega)i},\ \xi_2=\sqrt{(2Ek+\omega)i},\quad a=\frac{2iEk}{Pr-1},\ Pr\neq 1. \tag{23}$$

The components of the velocity field $u(z,t)$ and $v(z,t)$ are given by

$$u(z,t)=Re[q(z,t)],\ v(z,t)=Im[q(z,t)]. \tag{24}$$

It is important to point out the following properties of the solution given by Eq. (21):

a. $\lim_{s\to\infty}s\bar{q}(z,s)=0,\quad \lim_{t\to 0^+}q(z,t)=0,$

$$\lim_{s\to 0}s\bar{q}(z,s)=\frac{1-e^{-z\sqrt{2iEk}}}{2iEk},\ \lim_{t\to\infty}q(z,t)=\frac{1-e^{-z\sqrt{2iEk}}}{2iEk},$$

and, the functions

$$u_s(z)=\lim_{t\to\infty}u(z,t)=\frac{1}{2Ek}e^{-z\sqrt{Ek}}\sin(z\sqrt{Ek}),$$

$$v_s(z)=\lim_{t\to\infty}v(z,t)=\frac{e^{-z\sqrt{Ek}}\cos(z\sqrt{Ek})-1}{2Ek},$$

are the steady – state solutions.

b. $\lim_{z\to 0^+}\varphi(z,t;\alpha_1,\alpha_2)=2,\ \lim_{z\to 0^+}q(z,t)=\sin(\omega t),$

$$\lim_{z\to\infty}\bar{q}(z,t)=0,\ \lim_{z\to\infty}q(z,t)=0,\ \lim_{t\to 0}\varphi(z,t;\alpha_1,\alpha_2)=0.$$

2.2. Particular Case Pr = 1

As seen in the previous section, in the inversion of Laplace transforms was necessary to introduce condition $Pr\neq 1$. But, the getting solution for this case is very important for practical problems. It is known that, the Prandtl number is defined as the ratio of the molecular diffusivity of momentum to the molecular diffusivity of heat. It is a material property and it thus varies from fluid to fluid. For highly viscous oils, the Prandtl number is quiet large *(Pr >100*) and indicates rapid diffusion of momentum by viscous action compared to the diffusion of energy. In contrast, the liquid metals have very low values of Prandtl number, ($0.003 < Pr < 0.01$), that indicates more rapid diffusion of energy compared to the momentum diffusion rate. The Prandtl number for gases is near unity and, accordingly, the momentum and energy transfer by diffusion are comparable. The Prandtl number gives the link between velocity field and temperature field and its values strongly influence the relative growth of velocity and thermal boundary layers. If $Pr = 1$, then the thickness of velocity and thermal boundary layers have values of the same order of magnitude. In this case, Eq. (20) becomes

$$\bar{q}(z,s)=\frac{i}{2}\left(\frac{1}{s+\omega i}-\frac{1}{s-\omega i}\right)exp(-z\sqrt{s+2iEk})+\frac{i}{2Ek}\left(\frac{exp(-z\sqrt{s+2iEk})}{s}-\frac{exp(-z\sqrt{s})}{s}\right), \tag{25}$$

with the inverse Laplace transform

$$\begin{aligned}q_1(z,t)=&\frac{i}{4}e^{-i\omega t}\varphi(z,t;1,\xi_1)-\frac{i}{4}e^{i\omega t}\varphi(z,t;1,\xi_2)\\&+\frac{i}{4Ek}\varphi(z,t;1,\sqrt{2iEk})-\frac{i}{2Ek}erfc\left(\frac{z}{2\sqrt{t}}\right),\end{aligned} \tag{26}$$

and the velocity components

$$u_1(z,t)=Re[q_1(z,t)],\ v_1(z,t)=Im[q_1(z,t)]. \tag{27}$$

2.3 Homotopy PerturBation Method

In the previous section we have determined the inverse Laplace transform using the transformed velocity field $\bar{q}(z,s)$ in the form given by Eq. (20). Generally, the inverse of Laplace transform is difficult to compute by using the techniques of complex analysis. In (Madani *et al* (2011)), the couple Laplace transform-homotopy perturbation method is employed for solving one dimensional non-homogenous partial differential equations with a variable coefficient. The used method is a combination of the Laplace transform and the hmomotopy perturbation method (He (2006), Saberi-Nadjafi *et al* (2012)). In the present section we determine a new form of solution for the studied problem using the Laplace transform coupled with the homomtopy perturbation method. We reconsider the problem given by Eqs. (17), (18) and construct the homotopy

$$\Phi_p(z,s)=\bar{q}(z,s)-\frac{p}{s+2iEk}\left[\frac{\partial^2\bar{q}(z,s)}{\partial z^2}+\frac{e^{-z\sqrt{Prs}}}{s}\right]-(1-p)\frac{A}{s(s+2iEk)}e^{-z\sqrt{s+2iEk}}=0,\ p\in[0,1]. \tag{28}$$

Note that the process of varying the values of p from zero to unity corresponds to that of $\bar{q}(z,s)$ from

$\bar{q}_0(z,s)=\dfrac{A}{s(s+2iEk)}e^{-z\sqrt{s+2iEk}}$, $A=const.$, to

$\bar{q}(z,s)$ - solution of the problem given by Eqs. (17), (18). We assume that the solution $\bar{q}(z,s)$ can be written as a power series in p, that is

$$\bar{q}(z,s)=\sum_{k=0}^{\infty}p^k\bar{q}_k(z,s). \tag{29}$$

Substituting Eq. (29) in Eq. (28) and comparing the coefficients of powers of p, yields a successive procedure to determine functions $\bar{q}_k(z,s)$. Finally, by taking $p=1$ in Eq. (29), we obtain the solution of Eq.(17). The constant A is determined by using the boundary condition form Eq.$(18)_1$. So, we have the following relations:

$$p^0:\bar{q}_0(z,s)=\frac{A}{s(s+2iEk)}e^{-z\sqrt{s+2iEk}}, \tag{30}$$

$$p^1:\bar{q}_1(z,s)=\frac{1}{s+2iEk}\bar{q}_0''+\frac{e^{-z\sqrt{Prs}}}{s(s+2iEk)}-\frac{A}{s(s+2iEk)}e^{-z\sqrt{s+2iEk}}, \tag{31}$$

and, using Eq. (30), we obtain

$$\bar{q}_1(z,s)=\frac{1}{s(s+2iEk)}e^{-z\sqrt{Prs}}, \tag{32}$$

$$p^k:\bar{q}_k(z,s)=\frac{1}{s+2iEk}\bar{q}_{k-1}''(z,s), k\ge 2, k\in N, \tag{33}$$

where, the notation $\bar{q}_k''(z,s)=\dfrac{\partial^2\bar{q}_k(z,s)}{\partial z^2}$ has been used. From Eqs. (32) and (33) we get

$$\bar{q}_k(z,s)=\frac{(Prs)^{k-1}}{s(s+2iEk)^k}e^{-z\sqrt{Prs}}, \tag{34}$$

respectively, the solution of Eq. (17) in the form

$$\bar{q}(z,s)=\frac{A}{s(s+2iEk)}e^{-z\sqrt{s+2iEk}}+\sum_{k=1}^{\infty}\frac{(Prs)^{k-1}}{s(s+2iEk)^k}e^{-z\sqrt{Prs}}. \tag{35}$$

Function $\bar{q}(z,s)$ given by Eq. (35) satisfies Eq. $(18)_2$, namely, $\lim_{z\to\infty}\bar{q}(z,s)=0.$ Using the boundary condition from Eq. $(18)_1$, the final form of the solution is obtained as

$$\bar{q}(z,s)=\left[\frac{\omega}{s^2+\omega^2}-H(s)\right]e^{-z\sqrt{s+2iEk}}+H(s)e^{-z\sqrt{Prs}}, \tag{36}$$

where

$$H(s)=\sum_{k=1}^{\infty}\frac{(Prs)^{k-1}}{s(s+2iEk)^k}. \tag{37}$$

It is important to point out that function $\bar{q}(z,s)$ given by Eq. (36) is identical with that obtained in the previous section, namely with the function $q(z,s)$ given by Eq. (19). To show this, we rewrite H(s) in the following equivalent forms:

$$H(s)=\frac{1}{Prs}\sum_{k=1}^{\infty}\frac{1}{s}\left(\frac{Prs}{s+2iEk}\right)^k=\frac{1}{Prs}\sum_{k=1}^{\infty}\left(\frac{Pr}{2iEk}\right)^k\frac{1}{s}\left(\frac{1}{s}+\frac{1}{2iEk}\right)^{-k}. \tag{38}$$

Now, using the property

If $g(s)=L\{f(t)\}$ *then*

$$L^{-1}\left\{\frac{1}{s}g\left(\frac{1}{s}\right)\right\}=\int_0^{\infty}f(u)J_0(2\sqrt{tu})du, \tag{39}$$

we obtain that

$$\begin{aligned}h_1(t)&=L^{-1}\left\{\sum_{k=1}^{\infty}\left(\frac{\Pr}{2iEk}\right)^k\frac{1}{s}\left(\frac{1}{s}+\frac{1}{2iEk}\right)^{-k}\right\}\\&=\sum_{k=1}^{\infty}\left(\frac{\Pr}{2iEk}\right)^k\int_0^{\infty}\frac{u^{k-1}\exp\left(\frac{-u}{2iEk}\right)}{\Gamma(k)}J_0(2\sqrt{tu})du\\&=\int_0^{\infty}e^{\frac{-u}{2iEk}}J_0(2\sqrt{tu})\frac{\Pr}{2iEk}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{\Pr u}{2iEk}\right)^{k-1}du\\&=\frac{\Pr}{2iEk}\int_0^{\infty}\exp\left(\frac{(\Pr-1)u}{2iEk}\right)J_0(2\sqrt{tu})du.\end{aligned} \tag{40}$$

The expression given by Eq. (40) can be simplified by using the properties of Bessel functions (Watson (1995)). Finally, we obtain

$$h_1(t)=\frac{\Pr}{1-\Pr}\exp\left(\frac{2iEkt}{\Pr-1}\right), \tag{41}$$

respectively,

$$h(t)=L^{-1}\{H(s)\}=\frac{1}{2iEk}\left[1-\exp\left(\frac{2iEkt}{\Pr-1}\right)\right]. \tag{42}$$

Using Eq. (42) we obtain for the function H(s) the new form

$$H(s)=-\frac{1}{s[(\Pr-1)s-2iEk]}. \tag{43}$$

Replacing Eq. (43) in Eq. (36), we obtain the same form as that given by Eq. (19), therefore, functions from Eqs. (36) and (19) are identical. Applying the inverse Laplace transform to Eq. (36) we obtain the complex velocity field in the new forms, namely

$$q(z,t)=\frac{1}{2iEk}\int_0^t\left[2iEk\sin\omega(t-\tau)-1+\mathrm{e}^{\frac{2iEk(t-\tau)}{\mathrm{Pr}-1}}\right]\times\mathrm{e}^{-2iEk\tau-\frac{z^2}{4\tau}}\frac{z}{2\tau\sqrt{\pi\tau}}d\tau+\frac{1}{2iEk}\int_0^t\left[1-\mathrm{e}^{\frac{2iEk(t-\tau)}{\mathrm{Pr}-1}}\right]\mathrm{e}^{-\frac{z^2\mathrm{Pr}}{4\tau}}\frac{z\sqrt{\mathrm{Pr}}}{2\tau\sqrt{\pi\tau}}d\tau, \tag{44}$$

respectively,

$$q(z,t)=\frac{1}{iEk\sqrt{\pi}}\int_{\frac{z}{2\sqrt{t}}}^{\infty}\left[2iEk\sin\omega\left(t-\frac{z^2}{4x^2}\right)-1+\exp\left(\frac{2iEk}{Pr-1}\left(t-\frac{z^2}{4x^2}\right)\right)\right]\exp\left(-\left(x^2+\frac{z^2}{4x^2}\right)\right)dx+\int_{\frac{z\sqrt{Pr}}{2\sqrt{t}}}^{\infty}\left[1-\exp\left(\frac{2iEk}{Pr-1}\left(t-\frac{z^2Pr}{4x^2}\right)\right)\right]\frac{e^{-x^2}}{iEk\sqrt{\pi}}dx. \tag{45}$$

Obviously, the expressions given by Eqs. (44) and (45) are equivalent with the expression given by Eq. (21).

3. FRICTION COEFFICIENTS

In the studied problem we are interested of the shear stresses

$$\tau_x(z,t)=\mu\frac{\partial u(z,t)}{\partial z},\ \tau_y(z,t)=\mu\frac{\partial v(z,t)}{\partial z}. \tag{46}$$

By using dimensionless variables given by Eq. (7), the non-dimensional stresses

$$\tau_x^{'}=\frac{\tau_x}{\tau_0},\ \tau_y^{'}=\frac{\tau_y}{\tau_0},\ \tau_0=\rho U_0^2\sqrt{Gr}, \tag{47}$$

are given by (dropping the prime notation)

$$\tau_x(z,t)=\frac{\partial u(z,t)}{\partial z},\ \tau_y(z,t)=\frac{\partial v(z,t)}{\partial z}, \tag{48}$$

or, by the following complex stress field:

$$\tau(z,t)=\tau_x(z,t)+i\tau_y(z,t)=\frac{\partial q(z,t)}{\partial z}. \tag{49}$$

By applying Laplace transform to Eq. (49) and using Eq. (19), we obtain the image function

$$\bar{\tau}(z,s)=\frac{\partial\bar{q}(z,s)}{\partial z}=\frac{\sqrt{\mathrm{Pr}\,s}\,e^{-z\sqrt{\mathrm{Pr}\,s}}}{s[(\mathrm{Pr}-1)s-2iEk]}-\frac{\sqrt{s+2iEk}\,e^{-z\sqrt{s+2iEk}}}{s[(\mathrm{Pr}-1)s-2iEk]}-\frac{\omega\sqrt{s+2iEk}\,e^{-z\sqrt{s+2iEk}}}{s^2+\omega^2}. \tag{50}$$

The non-dimensional friction coefficient, in the transform domain is given by

$$\bar{C}_f(s)=\bar{\tau}(z,s)\big|_{z=0}=\frac{\sqrt{\mathrm{Pr}}}{\sqrt{s}}\frac{1}{(\mathrm{Pr}-1)s-2iEk}-\frac{\mathrm{Pr}}{(\mathrm{Pr}-1)s-2iEk}\frac{1}{\sqrt{s+2iEk}}+\frac{1}{s}\frac{1}{\sqrt{s+2iEk}}-\frac{\omega\sqrt{s+2iEk}}{s^2+\omega^2}. \tag{51}$$

Inverting the above image function, we find the complex form of friction coefficient

$$C_f(t)=\frac{\sqrt{Pr}}{Pr-1}\frac{e^{at}}{\sqrt{a}}erf(\sqrt{at})+\frac{erf(\sqrt{2iEkt})}{\sqrt{2iEk}}-\frac{Pr}{Pr-1}\frac{e^{at}}{\sqrt{a+2iEk}}erf(\sqrt{(a+2iEk)t})+\frac{\sqrt{(2Ek+\omega)i}}{2i}e^{i\omega t}erf(\sqrt{(\omega+2Ek)it})-\frac{\sqrt{(2Ek-\omega)i}}{2i}e^{-i\omega t}erf(\sqrt{(2Ek-\omega)it}). \tag{52}$$

The axial skin friction coefficient in the x-direction, respectively, y-direction are given by

$$C_{f_x}=\mathrm{Re}(C_f),\ C_{f_y}=\mathrm{Im}(C_f). \tag{53}$$

In the particular case $Pr = 1$, Eq. (51) becomes

$$\bar{C}_{f_1}(s)=\frac{i}{2Ek}\frac{1}{\sqrt{s}}-\frac{i}{2Ek}\frac{1}{\sqrt{s+2iEk}}+\frac{1}{s}\frac{1}{\sqrt{s+2iEk}}-\frac{\omega\sqrt{s+2iEk}}{s^2+\omega^2}, \tag{54}$$

and the complex friction coefficient is

$$C_{f_1}(t)=\frac{1-e^{-2iEkt}}{2Ek\sqrt{\pi t}}+\frac{erf(\sqrt{2iEkt})}{\sqrt{2iEk}}+\frac{\sqrt{(2Ek+\omega)i}}{2i}e^{i\omega t}erf(\sqrt{(2Ek+\omega)it})-\frac{\sqrt{(2Ek-\omega)i}}{2i}e^{-i\omega t}erf(\sqrt{(2Ek-\omega)it}), \tag{55}$$

with axial skin friction coefficients

$$C_{f_{1x}}=\mathrm{Re}(C_{f_1}),\ C_{f_{1y}}=\mathrm{Im}(C_{f_1}). \tag{56}$$

4. NUMERICAL RESULTS

In order to study the effects of the frame rotation on the fluid motion, the influence of the Ekman number on the velocity components $u(z, t)$ and $v(z, t)$ was studied by means of graphical illustrations. In Figs. 2 and 3 are plotted curves corresponding to velocity components $u(z, t)$ and $v(z, t)$, versus the spatial coordinate z, for $Pr = 0.75$, $\omega = 0.5$, $t\in\{0.5, 5, 15, 60\}$ and for small values, respectively, higher values of the Ekman number. From Figs. 2 a-f is observed that for small values of the Ekman number (so, for slow rotations of the frame), the absolute values of velocity components increase significantly in the vicinity of the plate. In points located away from the plate, the values of velocity components decrease and tend to zero.

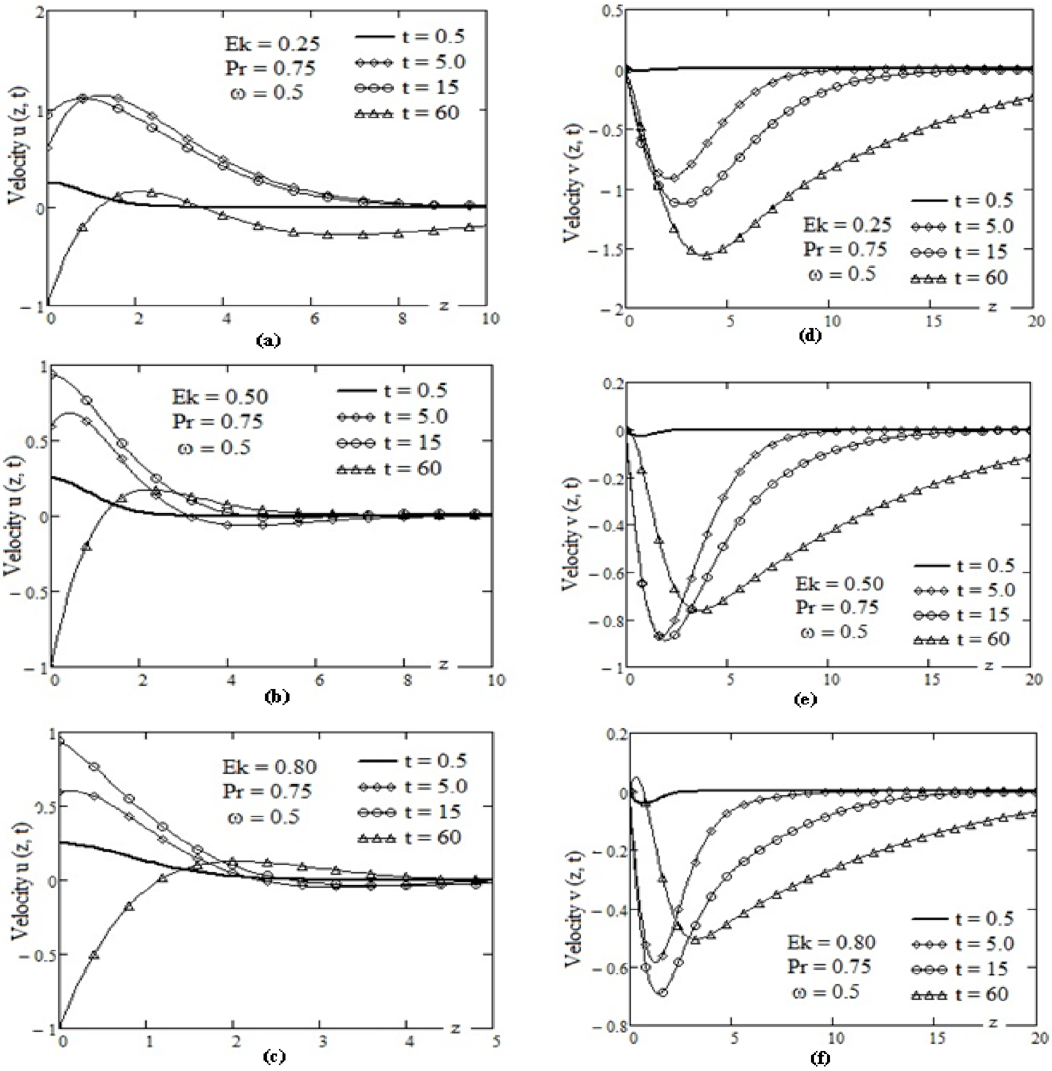

Fig. 2. Profiles of velocity components (small values of Ekman number).

Should also be noted that the velocity boundary layer thickness is less influenced by the increase of time but it is quite strongly influenced by increasing values of the Ekman number. Increasing of the Ekman number values leads to the reducing boundary layer thickness.

Figures 3 a-f were drawn for large values of Ekman number, so, for rotations of the frame with high angular velocity. Fluid behavior is similar to that shown by Figs. 2 with the difference that the boundary layer thickness shrinks significantly compared to the case of slow rotation. As shown in Figs. 3, the absolute values of velocity component along the y-axis direction are lower than those corresponding to small values of the Ekman number, while, the values of velocity component along the x-axis are comparable to the previous case.

The diagrams of the velocity components *u(z,t)* and *v(z,t)*, versus variable t, are sketched in Figs. 4 and 5. In these graphs we used the numerical values $\Pr = 0.75, \omega = 0.5, z \in \{0.25, 0.6, 1.0, 1.5\}$ and different values of Ekman number. From these figures it is observed that, for small or large values of Ekman number, the amplitudes of the velocity component *u(z,t)* decrease if the spatial coordinate *z* increases. In the case of subunit values of the Ekman number, amplitudes of the component *v(z,t)* increase with respect to *z* and are decreasing with respect to Ekman number. If the Ekman number is greater than 1, the velocity v(z,t) is decreasing with respect to the spatial coordinate z.

To plotting the curves shown in Figures 2-5, we used for the complex velocity field, the expressions given by Eqs. (21) and (44). The numerical calculations and graphical representations were carried out by subroutines existing in the Mathcad software. The numerical values presented in Table 1 reveals that the results are in excellent agreement.

5. CONCLUSION

In this paper, the unsteady flow of an

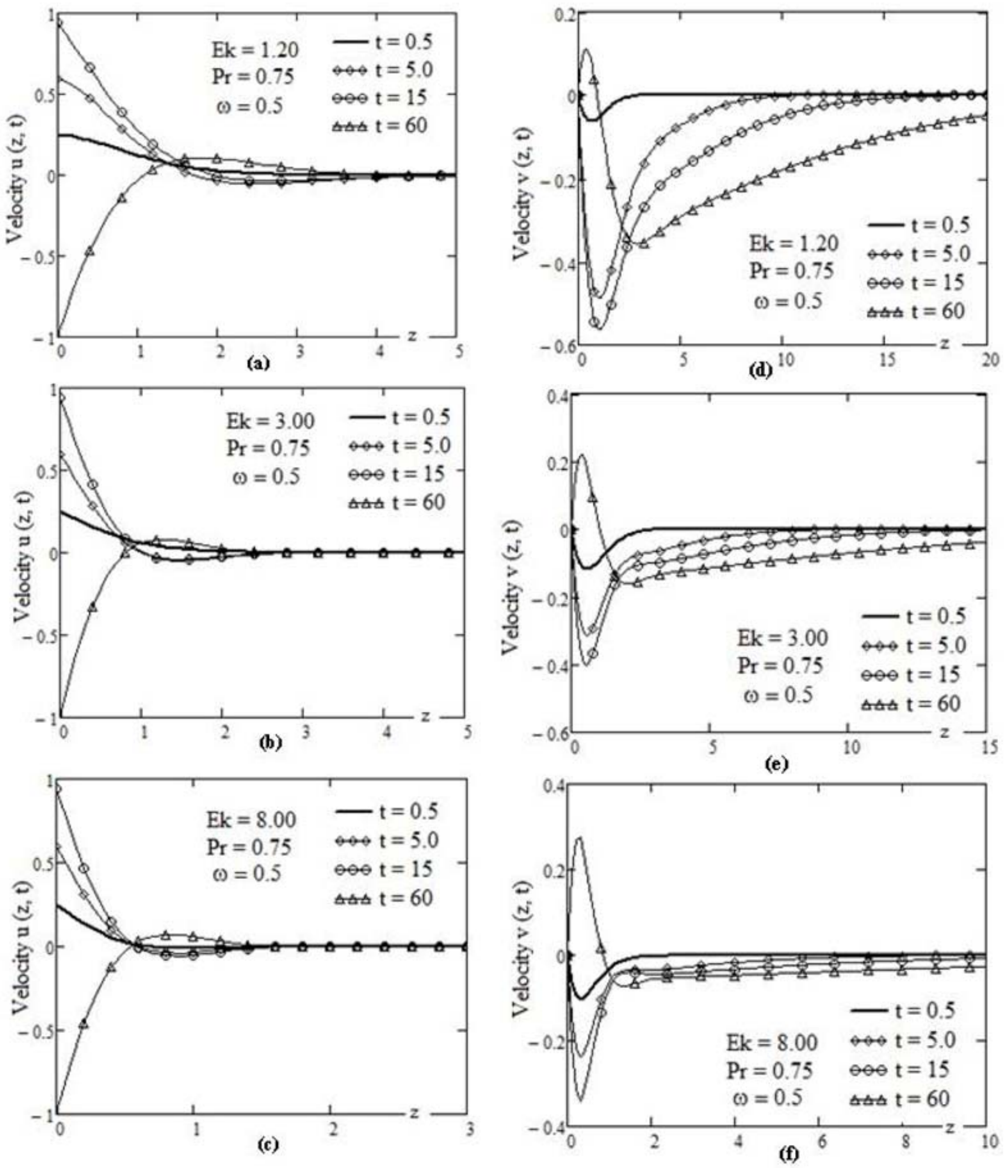

Fig. 3. Profiles of velocity components (large values of Ekman number).

Table 1 Comparison of values of the flow velocity for the present results (Laplace technique and homotopy perturbation method) with *t=5, Ek=0.9,Pr=0.7, ω=0.6*

z	*q(z, t)* – Eq. (21)	*q(z, t)* – Eq. (44)
0.1	0.190367-0.067691i	0.190366-0.067691i
0.2	0.225673-0.131532i	0.225673-0.131532i
0.3	0.248959-0.190526i	0.248959-0.190526i
0.4	0.262038-0.243979i	0.262037-0.243980i
0.5	0.266596-0.291455i	0.266596-0.291455i
0.6	0.264183-0.332735i	0.264183-0.332735i
0.7	0.256205-0.367782i	0.256205-0.367782i
0.8	0.243920-0.396707i	0.243920-0.396707i
0.9	0.228441-0.419737i	0.228441-0.419737i
1.0	0.210738-0.437189i	0.210738-0.437189i

incompressible, homogeneous Newtonian fluid past an oscillating isothermal vertical plate in a rotating frame was studied. The governing equations of the flow, together with initial and boundary conditions

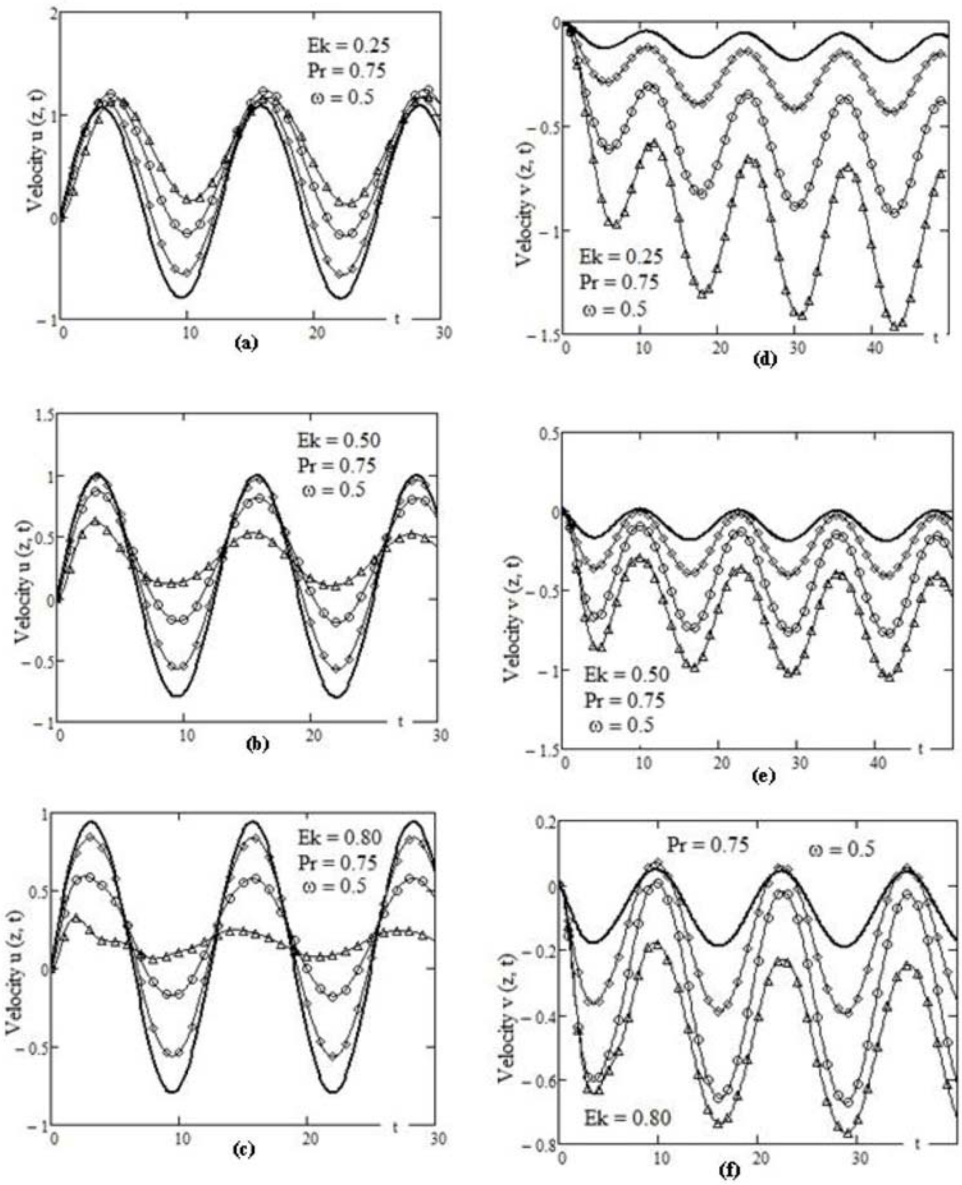

Fig. 4. Velocity diagrams for small values of Ekman number.

were written in the non-dimensional form and, for simplicity, a complex velocity field was used. By means of the Laplace transform method, respectively, the technique of coupling of the homotopy perturbation method with the Laplace transform method, two closed forms for the complex velocity field were determined. The equivalence of both forms was also demonstrated. The first method used to determine solution of the studied problem uses the Laplace transform with respect to time variable. The differential partial equation governing the motion of the fluid is transformed into a non-homogeneous ordinary differential equation with constant coefficients. Solution of this differential equation was determined by the classical method. It should be noted that, sometimes, the necessary calculations to determine solutions in the transform domain can be quite laborious or obtaining of the analytical solution is impossible. Also, obtaining of the inverse of Laplace transforms by means of the complex analysis techniques can be a difficult process.

The second method is based on the couple Laplace transform-homotopy perturbation method. In this case the solution of the ordinary differential equation which governing the flow is as a series. We must emphasize the fact that compared with the first method this pathway requires simpler calculations. Also, at least for the studied problem, the Laplace transforms inversion is easier. An important advantage of the method for the studied problem is that, the sum of the series solution was determined and so, was not necessary to approximate the solution with a partial sum of series. This method can be an adequate tool for solving nonlinear problems. In this case, possibly coupled with a numerical method of the Laplace transform inversion, this method can be a very

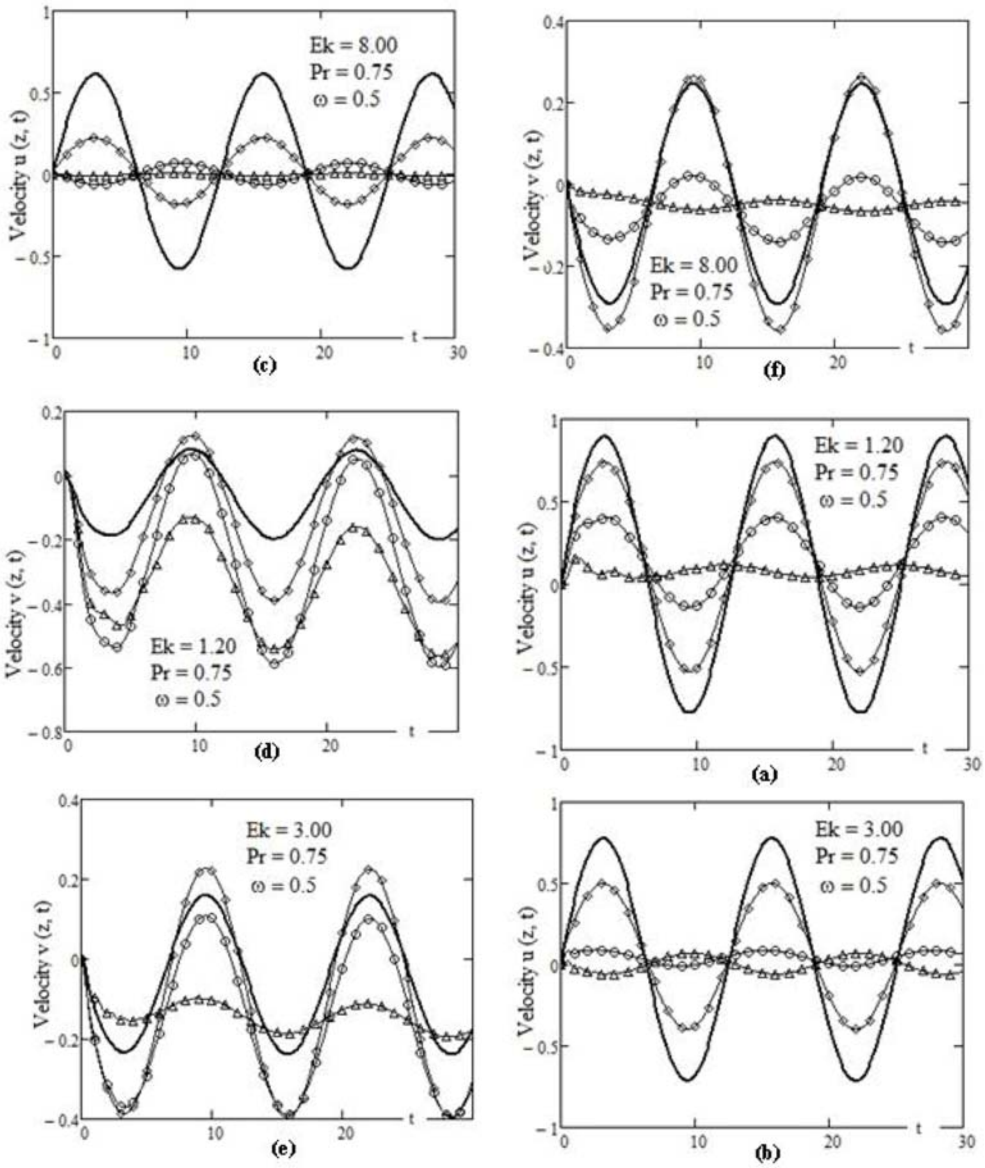

Fig. 5. Velocity diagrams for large values of Ekman number.

effective tool in the study of complex problems.Based on obtained solution and using the graphical illustrations generated with the Mathcad software, the influence of Ekman number on the velocity field was analyzed. It is obtained that, the faster frame rotations reduce the thickness of moving fluid layer.

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