Journal of Applied Fluid Mechanics, Vol. 8, No. 4, pp. 835-843, 2015.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.18869/acadpub.jafm.73.238.23258

Onset of the Mutual Thermal Effects of Solid Body and Nanofluid Flow over a Flat Plate Theoretical Study

A. Malvandi[1†], F. Hedayati[2], and D. D. Ganji[2]

[1] *Young Researchers and Elit Club. Qazvin Branch, Islamic Azad University, Iran*
[2] *Department of Mechanical Engineering, Islamic Azad University, Sari Branch, Sari, Iran*

†*Corresponding Author Email: a.Malvandi@qiau.ac.ir*

(Received May 23, 2014; accepted July 4, 2014)

ABSTRACT

The falling and settling of solid particles in gases and liquids is a natural phenomenon happens in many industrial processes. This phenomenon has altered pure forced convection to a combination of heat conduction and heat convection in a flow over a plate. In this paper, the coupling of conduction (inside the plate) and forced convection of a non-homogeneous nanofluid flow (over a flat plate) is investigated, which is classified in conjugate heat transfer problems. Two-component four-equation non-homogeneous equilibrium model for convective transport in nanofluids (mixture of water with particles<100nm) has been applied that incorporates the effects of the nanoparticles migration due to the thermophoresis and Brownian motion forces. Employing similarity variables, we have transformed the basic non-dimensional partial differential equations to ordinary differential ones and then solved numerically. Moreover, variation of the heat transfer and concentration rates with thermal resistance of the plate is studied in detail. Setting the lowest dependency of heat transfer rate to the thermal resistance of the plate as a goal, we have shown that for two nanofluids with similar heat transfer characteristics, the one with higher Brownian motion (lower nanoparticle diameter) is desired.

Keywords: Nanofluid; Flat plate; Conjugate heat transfer; Thermophoresis; Brownian motion.

NOMENCLATURE

a	Constant	p	pressure
C	nanoparticle volume fraction	Pr	Prandtl number
Cf	skin friction coefficient	q_w	surface heat flux
cp	specific heat at constant pressure	Re	Reynolds number
D_B	brownian diffusion coefficient	S	dimensionless suction/blowing parameter
D_T	thermophoresis diffusion coefficient	Sh	Sherwood number
Le	Lewis number	ρ	density

1. INTRODUCTION

There has been a significant amount of efforts to investigate the flow and heat transfer of a viscous incompressible fluid over a surface. After Blasius (1908) who studied boundary layer over a flat plate by employing a similarity transformation to reduce the partial differential boundary layer equations to a nonlinear third-order ordinary differential ones, a large amount of literatures on this issue have been conducted that are cited by Kays and Crawford (1980). The two well known boundary conditions in the mentioned literatures for energy equation named are constant wall temperature and constant heat flux. In these conditions, Thermal interactions between the fluid and the surface are particularly at the upper surface of the plate which gets wet by the fluid; however, in practical situations, the boundary condition at the lower surface are known and must be settled. Hence, heat transfer through the plate (conduction) must be taken into account in addition to the solid-fluid convective heat transfer. Not surprisingly, the effects of thermal resistance of the plate have to be included in the formulation, i.e. mutual thermal effects of fluid-solid have to be considered, which are usually referred as conjugated heat transfer problems.

A seminal study in this field was conducted by Perelman (1961) who studied boundary layer flow and heat transfer over a flat plate of finite thickness with two dimensional thermal conduction in the plate. Luikov et al. (1971) employed the Fourier sine transformation to solve the problem in terms of Fourier variables. Later, Luikov (1974) solved the same problem subject to the linear temperature distribution boundary condition in the plate. Then, a comprehensive survey on this subject conducted by Payvar (1977), Karvinen (1978), Pozzi and Lupo (1989) and Pop and Ingham (1993). It is worth mentioning that conjugated heat transfer has frequent industrial applications such as compact heat exchangers, solar collectors and coating materials particularly in turbine blades.

Along with the technology's improvement, enhancing the performance of conventional heat transfer became a main issue owing to low thermal conductivity of the most common fluids such as water, oil, and ethylene-glycol mixture. Since the thermal conductivity of solids is often higher than that of liquids, the idea of adding particles to a conventional fluid to enhance its heat transfer characteristics was emerged. Among all the dimensions of particles such as macro, micro, and nano, because of some obstacles in the pressure drop through the system or the problem of keeping the mixture homogeneous, nano-scaled particles have attracted more attention. These tiny particles are fairly close in size to the molecules of the base fluid and, thus, can realize extremely stable suspensions with slight gravitational settling over long periods. The word "nanofluid" was proposed by Choi (1995) to identify engineered colloids composed of nanoparticles dispersed in a base fluid. Following the seminal study of this concept by Masuda et al. (1993), a considerable amount of research in this field has risen exponentially. Meanwhile, theoretical studies emerged to model the nanofluid behaviors. To date, the proposed models are twofold: the homogeneous flow models and the dispersion models. Buongiorno (2006) indicated that the homogeneous models tend to under-predict the nanofluid heat transfer coefficient, and because of the nanoparticle size, the dispersion effect is completely negligible. Hence, Buongiorno developed an alternative model to explain the abnormal convective heat transfer enhancement in nanofluids and eliminate the shortcomings of the homogeneous and dispersion models. He considered seven slip mechanisms—the inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus, fluid drainage, and gravity—and claimed that, of these seven, only Brownian diffusion and thermophoresis are the important slip mechanisms in nanofluids. Moreover, Buongiorno concluded that the turbulence is not affected by nanoparticles. With this finding as basis, he proposed a two-component four-equation non-homogeneous equilibrium model for convective transport in nanofluids. Above-mentioned model has recently been used by Kuznetsov and Nield (2010) to study the influence of nanoparticles on natural convection boundary-layer flow past a vertical plate. Then, a comprehensive survey of convective transport of nanofluids in the boundary layer flow conducted by Sheikholeslami et al. (2012; 2013), Rashidi et al. (2014a, 2014b, and 2014c), Ganji and Malvandi (2014), Malvandi and Ganji (2014a, 2014b), Hassani et al. (2011), Malvandi et al. (2012; 2013), Bég et al. (2014), Ashorynejad et al. (2013), Soleimani et al. (2012) , Hatami et al. (2013), Malvandi (2013) and Alinia et al. (2011).

The focal point of this paper is to consider the impacts of the thermal resistance of the plate on the flow of a non-homogeneous nanofluid over a flat plate. Usual boundary layer equations along a flat plate have been considered for highlighting the effects of plate's thermal resistance. In contrast to simple heat boundary condition on upper surface of plate, the temperature at the lower surface is prescribed. Thus, the mutual thermal effects of conduction inside and convection along a flat plate have been considered. This problem is classified as conjugate heat transfer in nanofluid and to the best of the authors' knowledge, no investigation has been communicated so far. It must be declared that this problem can be considered as an extension (nanoparticles included) of Bejan (2004) and has applications in industries such as flat fins, cooling of electronic boards, solar collectors.

2. PROBLEM FORMULATION AND GOVERNING EQUATIONS

Consider the steady two-dimensional incompressible flow of a nanofluid over a semi-infinite flat plate which is heated by a highly convective fluid at beneath. Thickness of the plate *t(x)* is non-uniform, which may be varied along the plate's length, see Fig. 1, where for clarity, the variation of the thickness of the plate has been exaggerated.

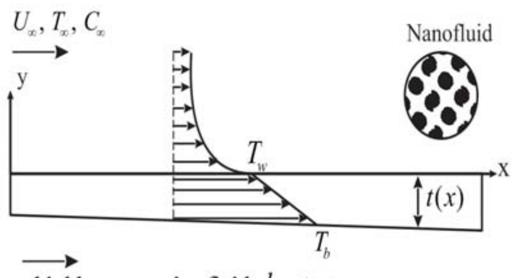


Fig. 1. The geometry of physical model and coordinate system.

The thickness of the plate is sufficiently smaller than its length; so it is reasonable to neglect the longitudinal conduction through the boundary layer and assume the linear temperature distribution in the plate. The nanofluid flows above the surface at a constant velocity, temperature and concentration of U_∞, T_∞, C_∞ respectively. The convective heat transfer coefficient for the beneath fluid is h_b which is great enough to maintain the lower surface at a constant temperature of T_b. Furthermore, the values of temperature and concentration at the top surface are named T_w, C_w respectively. This description can be a model for the case of solid coating Bejan (2004) or sedimentation in heat exchangers. Neglecting the effects of viscous dissipation on temperature gradients, the transport equations for mentioned problem including continuity, momentum and energy equations in the Cartesian coordinates can be expressed as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\tau\left[\begin{array}{l}D_B\left(\frac{\partial C}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right)\\+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial x}^2+\frac{\partial T}{\partial y}^2\right)\end{array}\right] \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

subject to the boundary conditions

$$\begin{gathered}u(x,y)=\begin{pmatrix}0 & y\to 0\\ U_\infty & y\to\infty\end{pmatrix},\quad v(x,0)=0\\ k_w\left.\frac{\partial T}{\partial y}\right)_{0-}=k\left.\frac{\partial T}{\partial y}\right)_{0+},\quad T(x,y\to\infty)=T_\infty\\ C(x,y\to 0)=C_w,\quad C(x,y\to\infty)=C_\infty\end{gathered} \tag{5}$$

It is apparent from Eq. (3) that heat transfer of a nanofluid is combined of conduction (first RHS term), convection (LHS term), and nanoparticle-diffusion (second RHS term). The LHS term of Eq. (4) indicates the slip velocity of nanoparticles relative to the base fluid, due to the combination of the Brownian motion (first RHS term) and the thermophoresis (second RHS term). For $t(x) << L$, the longitudinal conduction along the wall is negligible compared with the transverse conduction across the wall. Hence, assuming a linear distribution of temperature at the solid part, we can express the temperature gradient as follows

$$\left.\frac{\partial T}{\partial y}\right)_{0-}=\frac{T_w-T_b}{t(x)}$$

$$\begin{gathered}\psi=\sqrt{\nu x U_\infty}f(\eta),\ \eta=\sqrt{\frac{U_\infty}{\nu x}}y,\\ \phi=\frac{C-C_\infty}{C_w-C_\infty},\ \theta=\frac{T-T_\infty}{T_b-T_\infty}\end{gathered} \tag{6}$$

In addition, the physical properties including viscosity, thermal diffusivity, Prandtl number are assumed to be constant according to Buongiorno (2006), and MacDevette et al. (2014) . Next, Eqs. (1-4) can be reduced to the simpler ordinary differential equations by employing the similarity variables in the following form

Where ψ is the usual stream function, i.e. $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$, ν is the kinematic viscosity of the fluid. Substituting Eq. (7) into Eqs. (1)–(5), the following ordinary differential equation obtained

$$f'''+(1-\phi)^{2.5}\left(1-\phi+\phi\frac{\rho_s}{\rho_f}\right)\frac{f}{2}f''=0 \tag{7}$$

$$\frac{1}{Pr}\frac{k_s+2k_f-2\phi(k_f-k_s)}{k_s+2k_f+\phi(k_f-k_s)}\theta''+\left(\begin{array}{l}1-\phi+\\ \phi\frac{(\rho c_p)_s}{(\rho c_p)_f}\end{array}\right)\left(\frac{f}{2}\theta'+Nt\theta'^2+Nb\theta'\phi'\right)=0 \tag{8}$$

$$\phi''+\frac{f}{2}Le\phi'+\frac{Nt}{Nb}\theta'=0 \tag{9}$$

And the transformed boundary condition of Eq. (5) reduces to

$$\begin{gathered}f(0)=f'(0)=0,\ f'(\eta\to\infty)=1,\\ \theta(0)=1+J\theta'(0),\ \theta(\eta\to\infty)=0,\\ \phi(0)=1,\ \phi(\eta\to\infty)=0\end{gathered} \tag{10}$$

where ' denotes differentiation with respect to η and the outcoming non-dimensional parameters are

$$\begin{gathered}Pr=\frac{\nu}{\alpha},\quad Le=\frac{\nu}{D_B},\quad J=\frac{k_f}{k_w}\left(\frac{U_\infty t^2(x)}{\nu x}\right)^{\frac{1}{2}}\\ Nt=\frac{(\rho c_p)_p D_T(T_f-T_\infty)}{(\rho c_p)_f \nu T_\infty},\\ Nb=\frac{(\rho c_p)_p D_B(C_w-C_\infty)}{(\rho c_p)_f \nu}\end{gathered} \tag{11}$$

here Pr, Le, Nb, Nt, J represents the Prandtl number, the Lewis number, the Brownian motion, the thermophoresis and dimensionless plate resistance respectively, and the thermophysical properties of water/alumina nanoparticle and base fluid (water) are also provided as follows

$$\begin{gathered}c_{p_{bf}}=4182\text{J/(Kg K)},\ c_{p_p}=773\ \text{J/(Kg K)},\\ \rho_{bf}=998.2\ Kg/m^3,\ \rho_p=3380\ Kg/m^3,\\ k_{bf}=0.597,\ \text{W/(m K)},\ k_p=36\ \text{W/(m K)}\end{gathered}$$

$$\mu_{bf} = 9.93\times10^{-4}\ \text{Kg/(m s)} \qquad (12)$$

In General, J is a function of x which can be obtained from an energy balance at the surface, see Appendix. Existance of $J(x)$ makes it impossible to obtain the similarity solution; so, in order to remain focused on similarity solution, it has been assumed that the thickness of the plate varies in the form of $t(x)=\sqrt{x}$ to keep J constant. Needless to say, variations in thickness of the plate occur in the lower surface to avoid the inclination on the surface to keep the upper surface being flat. Pivotal quantities of interest including the skin friction coefficient, the local Nusselt and Sherwood numbers can be defined as

$$C_f = \frac{\tau_w}{\rho {u_\infty}^2},\quad Nu = \frac{xq_w}{K(T_b - T_\infty)},\quad Sh = \frac{xq_m}{D_B(C_w - C_\infty)} \qquad (13)$$

Where τ_w is the surface shear stress and, q_w and q_m are heat and mass flux at the surface respectively which are defined as follows

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},\ q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0},\quad q_m = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0} \qquad (14)$$

using the dimensionless variables Eq. (6), the rate of skin friction, heat transfer and concentration can be written as

$$Cfr = {Re_x}^{-1/2}C_f = f''(0),\quad \text{Nur} = {Re_x}^{-1/2}Nu = -\theta'(0),\quad \text{Shr} = {Re_x}^{-1/2}Sh = -\phi'(0) \qquad (15)$$

Here *Cfr, Nur, Shr* refers to reduced skin friction coefficient, the reduced Nusselt number and reduced Sherwood number respectively. Also, $Re_x = u_w x/v$ is the local Reynolds number base sheet's velocity. For considering the effects of J more precisely, the Nusselt and Sherwood numbers ratio can be defined as

$$\frac{Nu_j}{Nu_{j=0}} = \frac{\theta'(0)_j}{\theta'(0)_{j=0}},\quad \frac{Sh_j}{Sh_{j=0}} = \frac{\phi'(0)_j}{\phi'(0)_{j=0}} \qquad (16)$$

3. RESULTS AND DISCUSSIONS

The system of Eqs. (8-10) with boundary conditions of Eq. (11) have been solved numerically via Runge-Kutta-Fehlberg scheme. A Fortran code has been used to find the numerical solution of the present boundary value problem (BVP), the accuracy of which was shown elsewhere Malvandi et al. (2013) . Moreover, For our bulk computaions the far field boundary conditions denoted by η_{max} set to $\eta_{max}=10$ which was sufficient to achieve the far field boundary conditions asymptotically (shown later). In order to avoid the grid dependency, the integration step has been altered from 10^{-5} to 10^{-6} and there is no dependency was observed, as is shown in Table 1.

Table 1 Gird independent test when J=1 Nt=Nb=0.3, Le=5, Pr=7, and ϕ= 0.1.

dη	*-θ'(0)*	-□'(0)
10^{-5}	0.122822	0.698205
5×10^{-5}	0.122834	0.698212
10^{-6}	0.122843	0.698223

As it is clear, fluid mechanics part of the problem (Eq. (8)) is the famous Blasius problem which has been solved by many researchers and the results have been mentioned in several textbooks. For heat transfer term, substituting the $J = Nt = Nb = Le = 0$ in Eq. (9-10) the well known heat transfer over a flat plate was appeared. Here, the reported data of Kays and Crawford W.M. Kays and Crawford (1980) has been used in order to verify the developed code which has been shown in Table 2 and Fig. 2.

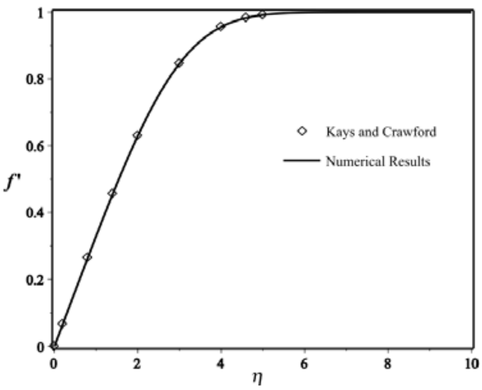


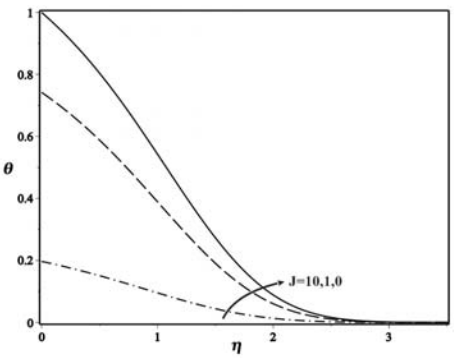
Fig. 2. Velocity boundary layer.

Table 2 Heat transfer rate for regular fluid for different values of prandtl number

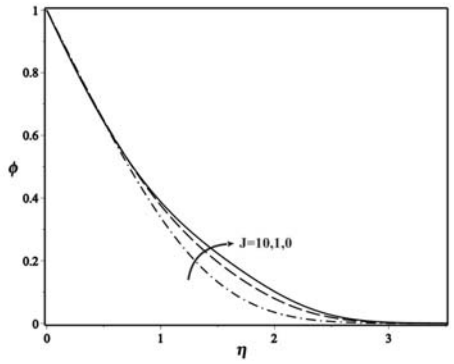
Pr	0.01	0.1	1	7	50
Kays and Crawford	0.05	0.140	0.332	0.645	1.24
Present study	0.05	0.140	0.332	0.645	1.24

The focal point of this study is the effects of J on the heat transfer characteristic of nanofluid flow over a flat plate. Calculations have been performed for the selective values: Nt, Nb, Le and the constant value of $\Pr = 7$ .Considering the effective parameters in the values of J such as k_f, ν, k_w , we have considered the range of 0.01 to 100 for J , see Bejan (2004) .

The effects of J on temperature and concentration profiles have been shown in Figs. 3a and 3b. These profiles essentially have the same form as regular fluids. The figures show that the solutions satisfy the far field boundary conditions asymptotically and can be used for the validity of our presumed computational domain. It is evident that an increase in J leads to a decrease in temperature and the concentration's profiles; however, the temperature profile is reduced more because, a rise in J increases the thermal resistance of the plate and consequently, the heat transfer rate decreases. Not surprisingly, the temperature at the plate declines as well.



(a)



(b)

Fig. 3. a) Temperature profile for different values of J , $Nt = Nb = 0.1, Le = 10$.
b) Concentration profile for different values of J , $Nt = Nb = 0.1$, $Le = 10$.

The heat transfer and concentration rates ratio versus J , for different values of Nt , Nb and Le have been demonstrated in Figs. 4-6. All the curves started from unity at $J = 0$ and have a same trend when J increases: an increase in J , decreases the ratio of heat transfer and concentration rates. The trend dwindles down when the value of J increases, which can be explained as follows. When $J = 0$ there is no thermal resistance at the plate and the dominated resistance is in the convective heat transfer above the plate. An increase in the value of J generates thermal resistance, which gradually increases the overall heat transfer resistance; not surprisingly, the heat transfer rate declines. This trend continues until the thermal resistance of the plate gets to the point (*J≈100*) where an increase in J has almost no effect on the heat transfer rate.

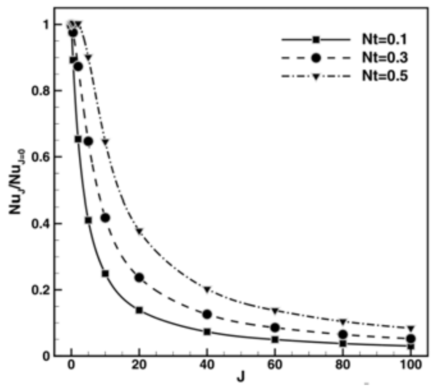
Figs. 4a and 4b show how the Lewis number affects the trends of the reduced Nusselt and Sherwood numbers' ratio versus J . The Lewis number defines the ratio of thermal diffusivity to mass diffusivity. It characterizes fluid flows where there is simultaneous heat and mass transfer by convection. Considering Fig. 4, we can state that decreasing the Lewis number increases the effects of J (the decreasing trend) on heat transfer and concentration rates. Therefore, it is found out that for two nanofluids, the one with a higher Lewis number experiences a lower reduction in the heat transfer rate. In other words, increasing Lewis number reduces the sensitivity of the heat transfer rate to the plate's thermal resistance J.

Studying the nano-sized particles, the effects of the Brownian motion have to be considered due to its significant effects on heat transfer and concentration rates. It should be stated that Brownian motion reflects the random drifting of suspended nanoparticles; on the other hand, thermophoresis is nanoparticle migration due to imposed temperature gradient across the fluid. The mentioned mechanisms are the two important slip mechanisms which appear as a result of nanoparticles' slip velocity to the base fluid. For hot surfaces, due to repelling the sub-micron sized particles, the thermophoresis tends to blow the nanoparticle volume fraction boundary layer away from the surface. Also, owing to the size of particles, the Brownian motion has a significant influence on the surrounding liquids. The effects of the thermophoresis number Nt , have been illustrated in Figs. 5a and 5b.

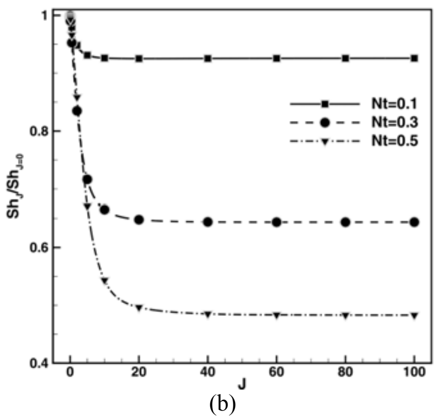
Evidently, increasing Nt leads to a rise in the heat transfer ratio and a fall in the concentration rate ratio. This is because as Nt increases, nanoparticle migration from the heated walls due to the thermophoresis increases. Thus, the value of nanoparticle concentration at the heated walls decreases which results in a fall in the concentration ratio. For $J < 20$, Nt has strong effects on the heat transfer rate; for example, when $J = 10$, the reduction of the heat transfer rate at $Nt = 0.5$ is approximately 35% whereas at $Nt = 0.1$ it is about 75%. However, the more J gets, the more suppression on the effects of Nt on the heat transfer rate occurs. Subsequently, while $J > 60$, the difference between the curves becomes insignificant. Figs. 6a and 6b signify the effects of the Brownian motion parameter, Nb, for the Nusselt and Sherwood numbers ratio. It can be observed that as Nb increases, both the ratio of the heat transfer and concentration rates increase.

This means that at higher values of Nb, J has insignificant effects on the heat transfer rate of the plate. Physically, Nb is higher for lower nanoparticle size ($Nb \propto 1/d_p$). So, reduction in the heat transfer rate due to the thernal resistance of the plate (J) is reduced for lower nanoparticles.

Thus, using lower nanoparticles enhance the performance of the system. Also, as Nb grows, regardless of the value of J, heat transfer rate declines. It can be seen that in Fig. 6a, even at the highest value of J, $J = 100$, an increase in Nb from 0.1 to 0.5 intensifies heat transfer rate significantly (37%). This outcome can be important in any systems which mechanisms such as sedimentations or solid-coating causes additional thermal resistance to the plate.

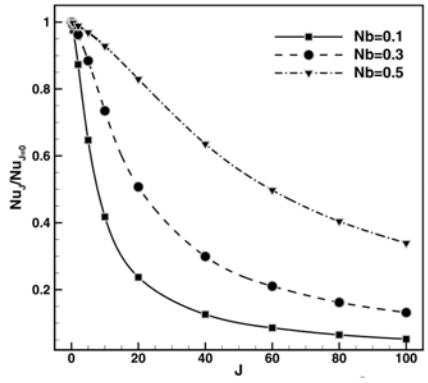


(a)

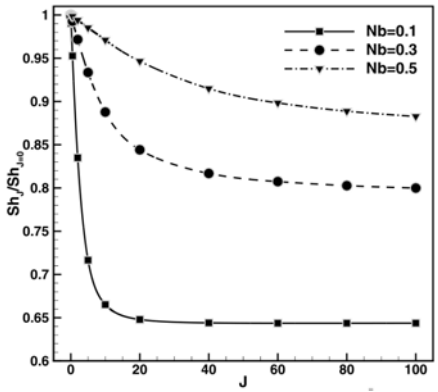


(b)

Fig. 4. a) Nusselt number ratio versus J for different values of Nt, $Le = 10$, $Nb = 0.1$.
b) Sherwood number ratio versus J for different values of Nt, $Le = 10$, $Nb = 0.1$.



(a)



(b)

Fig. 5. a) Nusselt number ratio versus *J* for different values of Nb, $Nt = 0.3, Le = 10$.
b) Sherwood number ratio versus *J* for different values of Nb, $Nt = 0.3, Le = 10$.

4. CONCLUSIONS

One knows how consequential is to prevent the heat transfer reduction originates from the thermal resistance of a system like sedimentations. Reducing the sensitivity of heat transfer behavior to thermal resistance of the plate as a goal, in this study, a pragmatic approach of boundary layer flow and heat transfer of nanofluid (mixture of water with particles<100nm) has been studied. This is classified onset of conjugate heat transfer problems. Employing similarity transformation, we have transformed the basic partial differential equations to ordinary differential ones before solving them numerically. The variation of heat and concentration rates with dimensionless plate thermal resistance J which arises from plate's thermal thickness is analyzed in details. Obtained results indicate that increasing *J* leads to decrease in both heat and concentration rates and among all parameters including Lewis number, Brownian motion and thermophoresis, it is shown that increasing in Brownian motion (lower nanoparticles size) may be the most effective way to suppress the effects of thermal resistance of the plate which reduces heat transfer rate. On the other hand, Lewis number has the least effect. Further, reduction in the heat transfer rate due to the thernal resistance of the plate (J) is reduced for lower nanoparticles. Also, using lower nanoparticles enhance the performance of the system. In addition, increasing in thermophoresis parameter for lower values of thermal resistance of the plate can decline the reduction of heat transfer due to sedimentations markedly.

REFERENCES

Alinia, M., D. D. Ganji and M. Gorji-Bandpy (2011). Numerical study of mixed convection in an inclined two sided lid driven cavity filled with nanofluid using two-phase mixture model. *International Communications in Heat and Mass Transfer* 38(10), 1428-1435

Ashorynejad, H. R., M. Sheikholeslami, I. Pop and D. D. Ganji (2013) Nanofluid flow and heat transfer due to a stretching cylinder in the presence of magnetic field. *Heat and Mass Transfer* 49 (3), 427-436.

Bég, O. A., M. M. Rashidi, M. Akbari and A. Hosseini (2014). Comparative numerical study of single-phase and two-phase models for bio-nanofluid transport phenomena. *Journal of Mechanics in Medicine and Biology* 14(1), 1-31.

Bejan, A. (2004). Convection heat transfer. Hoboken, N.J., Wiley.

Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z Math Phys.* 56, 1–37.

Buongiorno, J. (2006). Convective Transport in Nanofluids. *Journal of Heat Transfer* 128(3), 240-250.

Choi, S. U. S. (1995). Enhancing thermal conductivity of fluids with nanoparticles in Developments and Applications of Non-Newtonian Flows*, D. A. Siginer and H. P. Wang, Eds., ASME* 66, 99-105.

Ganji, D. D. and A. Malvandi (2014). Natural convection of nanofluids inside a vertical enclosure in the presence of a uniform magnetic field. *Powder Technology* 263, 50-57.

Hassani, M., M. Mohammad Tabar, H. Nemati, G. Domairry and F. Noori (2011). An analytical solution for boundary layer flow of a nanofluid past a stretching sheet. *International Journal of Thermal Sciences* 50(11), 2256-2263.

Hatami, M., R. Nouri and D. D. Ganji (2013). Forced convection analysis for MHD Al2O3–water nanofluid flow over a horizontal plate. *Journal of Molecular Liquids* 187, 294-301.

Karvinen, R. (1978). Note on conjugated heat transfer in a flat plate. *Letters in Heat and Mass Transfer* 5, 3–4, 197-202.

Karvinen, R. (1978). Some new results for conjugated heat transfer in a flat plate. *International Journal of Heat and Mass Transfer* 21, 9, 1261-1264.

Kays, W. M. and M. E. Crawford (1980). Convective Heat and Mass Transfer. New York, McGraw-Hill

Kuznetsov, A. V. and D. A. Nield (2010). Natural convective boundary-layer flow of a nanofluid past a vertical plate. *International Journal of Thermal Sciences* 49(2), 243-247.

Luikov, A. V. (1974). Conjugate convective heat transfer problems. *International Journal of Heat and Mass Transfer* 17(2), 257-265.

Luikov, A. V., V. A. Aleksashenko and A. A. Aleksashenko (1971). Analytical methods of solution of conjugated problems in convective heat transfer. *International Journal of Heat and Mass Transfer* 14(8), 1047-1056.

MacDevette, M. M., T. G. Myers and B. Wetton (2014). Boundary layer analysis and heat transfer of a nanofluid. *Microfluidics and Nanofluidics*, 1-12.

Malvandi, A. (2013). The Unsteady Flow of a Nanofluid in the Stagnation Point Region of a Time-dependent Rotating Sphere. *Thermal Science*, In Press .

Malvandi, A. and D. D. Ganji (2014). Brownian motion and thermophoresis effects on slip flow of alumina/water nanofluid inside a circular microchannel in the presence of a magnetic field. *International Journal of Thermal Sciences* 84, 196-206.

Malvandi, A. and D. D. Ganji (2014). Brownian motion and thermophoresis effects on slip flow of alumina/water nanofluid inside a circular microchannel in the presence of a magnetic field. *Journal of Magnetism and Magnetic Materials* 362, 172-179.

Malvandi, A. and D. D. Ganji (2014). Brownian motion and thermophoresis effects on slip flow of alumina/water nanofluid inside a circular microchannel in the presence of a magnetic field. *Powder Technology* 263, 37-44.

Malvandi, A. and D. D. Ganji (2014). Brownian motion and thermophoresis effects on slip flow of alumina/water nanofluid inside a circular microchannel in the presence of a magnetic field. *International Journal of Thermal Sciences* 84(4), 196-206.

Malvandi, A., D. D. Ganji, F. Hedayati and E. Yousefi Rad (2013). An analytical study on entropy generation of nanofluids over a flat plate. *Alexandria Engineering Journal* 52(4), 595-604.

Malvandi, A., F. Hedayati and G. Domairry (2013). Stagnation point flow of a nanofluid toward an exponentially stretching sheet with nonuniform heat generation/absorption. *Journal of Thermodynamics*.

Malvandi, A., F. Hedayati and D. D. Ganji (2013). Thermodynamic optimization of fluid flow over an isothermal moving plate. *Alexandria Engineering Journal* 52(3), 277-283.

Malvandi, A., F. Hedayati and D. D. Ganji (2014). Slip effects on unsteady stagnation point flow of a nanofluid over a stretching sheet. *Powder Technology* 253(0) 377-384.

Malvandi, A., F. Hedayati and M. R. H. Nobari (2014). An HAM Analysis of Stagnation-Point Flow of a Nanofluid over a Porous Stretching Sheet with Heat Generation. *Journal of Applied Fluid Mechanics* 7(1) 135-145.

Malvandi, A., F. Hedayati and M. R. H. Nobari (2014). An Analytical Study on Boundary Layer Flow and Heat Transfer of Nanofluid Induced by a Non-Linearly Stretching Sheet. *Journal of Applied Fluid Mechanics* 7(2), 375-384.

Malvandi, A., F. Hedayeti, D. Ganji and Y. Rostamiyan (2013). Unsteady boundary-layer flow of nanofluid past a permeable stretching/shrinking sheet with convective heat transfer. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science* 228(7), 1175-1184.

Malvandi, A., S. A. Moshizi, E. G. Soltani and D. D. Ganji (2014). Modified Buongiorno's model for fully developed mixed convection flow of nanofluids in a vertical annular pipe. *Computers & Fluids* 89(0) 124-132.

Masuda, H., A. Ebata, K. Teramae and N. Hishinuma (1993). Alteration of thermalconductivity and viscosity of liquid by dispersing ultra-fine particles. *Netsu Bussei* 7, 4.

Payvar, P. (1977). Convective heat transfer to laminar flow over a plate of finite thickness. *International Journal of Heat and Mass Transfer* 20(4), 431-433.

Perelman, T. L. (1961). On conjugated problems of heat transfer. *International Journal of Heat and Mass Transfer* 3(4), 293-303.

Pop, I. and D. B. Ingham (1993). A note on conjugate forced convection boundary-layer flow past a flat plate. *International Journal of Heat and Mass Transfer* 36(15), 3873-3876.

Pozzi, A. and M. Lupo (1989). The coupling of conduction with forced convection over a flat plate. *International Journal of Heat and Mass Transfer* 32(7), 1207-1214.

Rashidi, M. M., A. Hosseini, I. Pop, S. Kumar and N. Freidoonimehr (2014). Comparative numerical study of single and two-phase models of nanofluid heat transfer in wavy channel. *Applied Mathematics and Mechanics,* 1-18.

Rashidi, M. M., E. Momoniat, M. Ferdows and A. Basiriparsa (2014). Lie Group Solution for Free Convective Flow of a Nanofluid Past a Chemically Reacting Horizontal Plate in a Porous Media. *Mathematical Problems in Engineering.*

Rashidi, M. M., N. Freidoonimehr, A. Hosseini, O. Anwar Bég and T-K. Hung (2014). Homotopy simulation of nanofluid dynamics from a non-linearly stretching isothermal permeable sheet with transpiration. *Meccanica* 49(2), 469-482.

Sheikholeslami, M., F. Bani Sheykholeslami, S. Khoshhal, H. Mola-Abasia, D. D. Ganji and H. Rokni (2013). Effect of magnetic field on Cu–water nanofluid heat transfer using GMDH-type neural network. *Neural Computing and Applications*, 1-8.

Sheikholeslami, M., D. D. Ganji and H. R. Ashorynejad (2013). Investigation of squeezing unsteady nanofluid flow using ADM. *Powder Technology* 239(0) 259-265.

Sheikholeslami, M., D. D. Ganji, H. R. Ashorynejad and H. B. Rokni (2012). Analytical investigation of Jeffery-Hamel flow with high magnetic field and nanoparticle by Adomian decomposition method. *Applied Mathematics and Mechanics* 33(1), 25-36.

Sheikholeslami, M., M. Gorji-Bandpay and D. D. Ganji (2012). Magnetic field effects on natural convection around a horizontal circular cylinder inside a square enclosure filled with nanofluid. *International Communications in Heat and Mass Transfer* 39(7), 978-986.

Sheikholeslami, M., M. Gorji-Bandpy and D. D. Ganji , S. Soleimani Natural convection heat transfer in a cavity with sinusoidal wall filled with CuO–water nanofluid in presence of magnetic field. *Journal of the Taiwan Institute of Chemical Engineers*, 0.

Sheikholeslami, M., M. Gorji-Bandpy, D. D. Ganji, S. Soleimani and S. M. Seyyedi (2012). Natural convection of nanofluids in an enclosure between a circular and a sinusoidal cylinder in the presence of magnetic field. *International Communications in Heat and Mass Transfer* 39(9), 1435-1443.

Sheikholeslami, M., M. Gorji-Bandpy, S. M. Seyyedi, D. D. Ganji, H. B. Rokni and S. Soleimani (2013). Application of LBM in simulation of natural convection in a nanofluid filled square cavity with curve boundaries. *Powder Technology* 247(0), 87-94.

Sheikholeslami, M., M. Hatami and D. D. Ganji (2013). Analytical investigation of MHD nanofluid flow in a semi-porous channel. *Powder Technology* 246(0), 327-336.

Soleimani, S., M. Sheikholeslami, D. D. Ganji and M. Gorji-Bandpay (2012). Natural convection heat transfer in a nanofluid filled semi-annulus enclosure. *International Communications in Heat and Mass Transfer* 39(4), 565-574.

Appendix

With in mind the following similarity variables (Eq. (7))

$$\eta = \sqrt{\frac{U_\infty}{\nu x}}\, y,\ \theta = \frac{T - T_\infty}{T_b - T_\infty} \qquad \text{(A-1)}$$

Energy balance equation for any control volume at the surface is:

$$k \left.\frac{\partial T}{\partial y}\right)_w = k_w \frac{T_w - T_b}{t(x)} \qquad \text{(A-2)}$$

Employing chain rule in derivation

$$k \left.\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\right)_w = k_w \frac{T_w - T_b}{t(x)} \qquad \text{(A-3)}$$

Applying dimensionless parameter $\theta = \frac{T - T_\infty}{T_b - T_\infty}$ we have

$$(T_b - T_\infty) k \left.\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}\right)_w = k_w \frac{T_w - T_b}{t(x)} \qquad \text{(A-4)}$$

with some simplifications

$$\left(\frac{k\eta_y}{k_w} t(x)\right)\frac{\partial \theta}{\partial \eta} = \frac{T - T_\infty + T_\infty - T_b}{T_b - T_\infty} = \theta - 1 \qquad \text{(A-5)}$$

with substituting the $\eta_y = \sqrt{\frac{U_\infty}{\nu x}}$ we may have

$$\left(\underbrace{\frac{k_f}{k_w}\left(\frac{U_\infty t^2(x)}{\nu x}\right)^{\frac{1}{2}}}_{J}\right)\frac{\partial \theta}{\partial \eta} = \theta - 1 \qquad \text{(A-6)}$$

So the surface condition can be expressed as

$$\theta = 1 + J(x)\theta' \qquad \text{(A-7)}$$