Mixed Convection Falkner-Skan Wedge Flow of an Oldroyd-B Fluid in Presence of Thermal Radiation

M. Bilal Ashraf†, T. Hayat2,3, and H. Alsulami3

1 Department of Mathematics, Comsats Institute of Information Technology WahCantt. 47040, Pakistan
2 Department of Mathematics, Quaid-i-Azam University 45320 Islamabad 44000, Pakistan
3 Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

†Corresponding Author Email: bilalashraf_qau@yahoo.com

(Received December 11, 2014; accepted February 9, 2015)

ABSTRACT

The present study deals with the Falkner-Skan flow of rate type non-Newtonian fluid. Expressions of an Oldroyd-B fluid in the presence of mixed convection and thermal radiation are used in the development of relevant equations. The resulting partial differential equations are reduced into the ordinary differential equations employing appropriate transformations. Expressions of flow and heat transfer are constructed. Convergence of derived nonsimilar series solutions is guaranteed. Impact of various parameters involved in the flow and heat transfer results is plotted and examined.

Keywords: Oldroyd-B fluid; Mixed convection; Thermal radiation; Falkner-Skan flow.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cρ</td>
<td>specific heat</td>
</tr>
<tr>
<td>g</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>Grx</td>
<td>the local Grashof number</td>
</tr>
<tr>
<td>κT</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>Nu</td>
<td>local Nusselt number</td>
</tr>
<tr>
<td>Pr</td>
<td>Prandtl number</td>
</tr>
<tr>
<td>R</td>
<td>radiation parameter</td>
</tr>
<tr>
<td>ReX</td>
<td>is the local Reynold number</td>
</tr>
<tr>
<td>u and v</td>
<td>are velocity components</td>
</tr>
<tr>
<td>μ</td>
<td>is dynamic viscosity</td>
</tr>
<tr>
<td>ρ</td>
<td>density of fluid</td>
</tr>
<tr>
<td>α</td>
<td>wedge angle</td>
</tr>
<tr>
<td>λ1</td>
<td>is the relaxation time,</td>
</tr>
<tr>
<td>λ2</td>
<td>is the retardation time,</td>
</tr>
<tr>
<td>λ</td>
<td>mixed convection parameter</td>
</tr>
<tr>
<td>β1 and β2</td>
<td>are the dimensionless parameters</td>
</tr>
</tbody>
</table>

1. INTRODUCTION

The non-Newtonian fluids are encountered in the industrial and technological applications. Many fluids in nature do not obey the Newton's law of viscosity. Such fluids include polymer solutions, colloidal and suspension solutions, apple sauce, clay coating, shampoos, paints, certain oils, cement, sludge, drilling muds, food products, paper pulp, aqueous foams, slurries, grease etc. The rheological characteristics of non-Newtonian fluids cannot be predicted by employing a single relationship between stress and shear rate. Hence the non-Newtonian fluids are mainly classified into three types namely, the integral, differential and rate. Extensive studies in the literature have been devoted to the flows of differential type fluids in various geometries. However the differential type fluids are insufficient to describe the effects of relaxation and retardation times although they own the properties of normal stress, shear thinning and shear thickening. The rate type fluids are significant for the description of relaxation and retardation times effects. Oldroyd-B model is known as one of the subclass of rate type models which can predict the effects of relaxation and retardation times. Some of the recent studies relevant to Oldroyd-B model include the researches of (Fetecau et al. 2010, Fetecau et al. 2009, Zheng et al. 2011, Haitao et al. 2009a, Haitao et al. 2009b, Liu et al. 2011, Zheng et al. 2011, Fetecau et al. 2011, Jamil et al. 2011, Hayat et al. 2012).

In the field of aerodynamics, the analysis of two-dimensional boundary layer problems for steady
and incompressible flow passing a wedge is common area of interest. The study of heat transfer along a wedge has gained considerable attention due to its vast applications in industry and its important bearings in several technological and natural processes. Falkner and Skan (1931) firstly discussed the momentum boundary layer equation of two-dimensional wedge flows. They proposed a similarity transformation method in order to reduce the partial differential equation to the non-linear third order ordinary differential equation. Afterwards Hartree (1937) found the solution of that problem and also obtained the numerical values of the wall shear stress for different values of wedge angle. Abbasbandy and Hayat (2009) employed Hankel-Pade method to calculate the skin friction coefficients of the MHD Falkner-Skan boundary layer flow of viscous fluid. In another article, Abbasbandy and Hayat (2009) used the homotopy analysis method to obtain the analytic solution to this problem. Prandtl et al. (2011) established the MHD Falkner-Skan flow on a fixed and impermeable wedge by using the pseudospectral method of the Hermite functions. Kuo (2005) employed the differential transform method to analyze the heat transfer in the Falkner-Skan wedge flow. Radiative effects have important applications in several problems of physics and engineering. The radiation heat transfer effects in different flows are important in space technology and high temperature process. But a very little is known about the effects of radiation in the boundary layer flow of rate type fluids. Thermal radiation effects may play an important role in controlling heat transfer in polymer processing industry where the quality of the final product depends on the heat controlling factors to some extent. High temperature plasmas, cooling of nuclear reactors, liquid metal fluids, power generation systems are some important applications of radiative heat transfer from a wall to conductive gray fluids. A very significant area of research in radiative heat transfer, at the present time is the numerical simulation of combined radiation and convection / conduction transport processes. The effort has arisen largely due to the need to optimize industrial system such as furnaces, ovens and boilers and the interest in our environment and in no conventional energy sources, such as the use of salt gradient solar ponds for energy collection and storage. In particular, mixed convection induced by the simultaneous action of buoyancy forces resulting from thermal diffusion is of considerable interest in nature and in many industrial applications such as geophysics, oceanography, drying processes, solidification of binary alloy and chemical engineering. The effect of radiation in heat transfer problems have been studied by Chen (2009), Mukhopadhyay (2009) and Hayat et al. (2010). Recently, Kim (2001) developed the numerical treatment for the Falkner-Skan wedge flow of a power law fluid. Hayat et al. (2011) extended the analysis of Kim (2001) for mixed convection. Ishak (2010) presented the similar solutions for flow and heat transfer over a permeable surface with convective boundary conditions.

The aim of present paper is to analyze the radiative Falkner-Skan flow of an Oldroyd-B fluid in the presence of mixed convection. The presentation of article is made as follows. Problem is formulated in the next section. Section three consists of the series solutions of the governing problem by using a very useful technique namely the homotopy analysis method (Liao (2012), Liu (2013), Hayat et al. (2013), Abbasbandy et al. (2013), Zheng et al. (2012), Rashidi et al. (2014), Turkylmazoglu (2012), Ashraf et al. (2015)). Convergence analysis and the impact of various parameters of interest are presented in section four. The last section includes the main observations.

2. PROBLEM DEVELOPMENT

Let us investigate the two-dimensional Falkner-Skan flow of an Oldroyd-B fluid. We further consider the heat transfer. Cartesian coordinates \((x,y)\) are used such that \(x\) -axis is parallel to the wall and \(y\) -axis normal to it. An incompressible fluid occupies the region \(y \geq 0\). The equations governing the present flow situation are based on the conservation laws of mass, linear momentum and energy. Flow diagram of the problem is as follows:

Taking into account the aforementioned assumptions, the resulting boundary layer equations can be written as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
u \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \lambda_1 \left( u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right) + \nu \lambda_2 \left( \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \right) + \frac{\alpha}{2} \lambda_2 \left( \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 v}{\partial y^3} \right) = \frac{\alpha}{2} \lambda_1 \left( \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 v}{\partial y^3} \right) \quad (2)
\]

\[
\frac{\rho c_p}{\gamma} \left( \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{16 \sigma T^3}{3k} + k_T \right) \frac{\partial T}{\partial y} \quad (3)
\]

The appropriate boundary conditions are

\[
\begin{align*}
  u &= U, & v &= 0, & T &= T_w = T_\infty + Ax^\xi \\
  &\text{at } y = 0, & u &\to 0, & T &\to T_\infty \text{ as } y \to \infty 
\end{align*} \quad (4)
\]

where \(U = ax^n\) is the free stream velocity, \(\mu\) is the dynamic viscosity, \(\lambda_1\) is the relaxation time, \(\lambda_2\)
is the retardation time, \( \alpha \) is the wedge angle, \( k_T \) is the thermal conductivity, \( k \) is the surface temperature exponent, \( T \) and \( T_w \) are the temperatures of the fluid and ambient respectively and \( T_w \) is the wall temperature. We utilize [13, 20, 21]

\[
u = \frac{n+1}{2} \sqrt{U x y},
\]

\[
\psi = \frac{2}{n+1} \sqrt{v x} U (\eta),
\]

\[
v = -\frac{n+1}{2} \sqrt{\frac{U}{x}} [f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)],
\]

\[
\theta(\eta) = \frac{T - T_w}{T_w - T_w},
\]

where \( \eta \) is the similarity variable, \( v \) is the stream function, \( f \) is the dimensionless stream function and \( \theta \) is the dimensionless temperature. Now the continuity equation (1) is identically satisfied and Eqs. (2) - (4) leads

\[
f^{**} + f^* + \beta_1 f^{**} + \beta_2 f^{**} + \beta_3 f^{**} = \frac{2n}{n+1} \frac{\alpha}{\eta} + \frac{2n}{n+1} \frac{\partial \sin \alpha}{\partial \eta} = 0,
\]

where \( \beta_1, \beta_2, \text{and} \beta_3 \) are the non-zero auxiliary parameters then the zeroth-order and \( m \) th order deformation problems are stated as follows.

\[
(1-p)L_f \left[ \hat{f}(\eta; p) \right] = p h_f \left[ f(\eta; p), \hat{f}(\eta; p) \right],
\]

\[
(1-p)L_\theta \left[ \hat{\theta}(\eta; p) \right] = p h_\theta \left[ \theta(\eta; p), \hat{\theta}(\eta; p) \right],
\]

\[
\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 0, \quad \hat{f}'(\infty; p) = 1,
\]

\[
\hat{\theta}(0; p) = 0, \quad \hat{\theta}(\infty; p) = 0,
\]

which in dimensionless form gives

\[
(Re_x)^{1/2} Nu_x = -\theta(0).
\]

Series solutions

The initial guesses \( f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}, \)

\[
L_f \left[ f(\eta) \right] = \frac{d^2 f}{d\eta^2} + \frac{df}{d\eta}, \quad L_\theta(\theta) = \frac{d^2 \theta}{d\eta^2} - \theta,
\]

With

\[
L_f \left[ C_1 + C_2 \exp(\eta) \right] + C_3 \exp(-\eta) = 0,
\]

\[
L_\theta[C_4 \exp(\eta) + C_5 \exp(-\eta)] = 0,
\]

where \( C_i \) (i = 1 - 5) are the arbitrary constants. If \( p \in [0,1] \) is the embedding parameter and \( h_f \) and \( h_\theta \) are the non-zero auxiliary parameters then the zeroth-order and \( m \) th order deformation problems are stated as follows.
\[
\begin{align*}
\mathbf{R}^f_m (\eta) &= f_m (\eta) + \sum_{k=0}^{m-1} \left[ \beta_k \left( -\frac{2n}{n+1} f_{m-1-k} f_k - \frac{3n+1}{2} f_{m-1-k} f_k \right) \right] \\
&\quad + \beta_2 \left( -\frac{2n}{n+1} f_{m-1-k} f_k - \frac{3n+1}{2} f_{m-1-k} f_k \right) \\
&\quad + \frac{2}{n+1} \lambda \theta_{m-1} \sin \frac{\alpha}{2}, \\
\mathbf{R}^\theta_m (\eta) &= \theta_m + \sum_{k=0}^{m-1} \left[ \theta_k \left( -\frac{2n}{n+1} f_{m-1-k} f_k - \frac{3n+1}{2} f_{m-1-k} f_k \right) \right] \\
&\quad + \beta_2 \left( -\frac{2n}{n+1} f_{m-1-k} f_k - \frac{3n+1}{2} f_{m-1-k} f_k \right) \\
&\quad + \frac{2}{n+1} \lambda \theta_{m-1} \sin \frac{\alpha}{2},
\end{align*}
\]

\begin{align*}
\bar{f} (\eta; 0) &= f_0 (\eta), \quad \bar{f} (\eta; 1) = f (\eta), \\
\bar{\theta} (\eta; 0) &= \theta_0 (\eta), \quad \bar{\theta} (\eta; 1) = \theta (\eta), \\
\text{and when } p \text{ increases from 0 to 1 then } \bar{f} (\eta; p) \text{ and } \bar{\theta} (\eta; p) \text{ vary from the initial solutions } f_0 (\eta) \text{ and } \theta_0 (\eta) \text{ to final solutions } f (\eta) \text{ and } \theta (\eta) \text{ respectively. By Taylor's expansion one has}
\end{align*}

\begin{align*}
\bar{f} (\eta; p) &= f_0 (\eta) + \sum_{m=1}^{\infty} f_m (\eta) p^m, \\
\bar{\theta} (\eta; p) &= \theta_0 (\eta) + \sum_{m=1}^{\infty} \theta_m (\eta) p^m,
\end{align*}

where the auxiliary parameters are so properly chosen that the series (32) and (33) converge at \( p = 1 \) i.e.

\begin{align*}
f (\eta) &= f_0 (\eta) + \sum_{m=1}^{\infty} f_m (\eta), \\
\theta (\eta) &= \theta_0 (\eta) + \sum_{m=1}^{\infty} \theta_m (\eta).
\end{align*}

The general solutions are

\begin{align*}
f_m (\eta) &= f_m (\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \\
\theta_m (\eta) &= \theta_m (\eta) + C_4 e^\eta + C_5 e^{-\eta},
\end{align*}

In which \( f_m (\eta) \) and \( \theta_m (\eta) \) are special functions.

Convergence of the series solutions

Note that the series solutions in Eqs. (34) and (35) contain two auxiliary parameters \( h_f \) and \( h_\theta \).

The convergence of series solutions depend upon these auxiliary parameters. For range of values of these parameters, the \( h \)-curves at 15-th-order of approximations have been plotted in Fig. 2. It is found that the admissible values of \( h_f \) and \( h_\theta \) are \(-1.3 \leq h_f \leq -0.25 \) and \(-1.2 \leq h_\theta \leq -0.5 \). The series converge in the whole region of \( \eta \) when \( h_f = h_\theta = -1.0 \). The residual errors of \( f \) and \( \theta \) are
Fig. 1. Physical Model.

Residual error of
\[ f = f_m + f_m f_m' \beta (2n+1) \left( 2 - \frac{n+1}{n+1} \right) \]
\[ + \left( 3n + 1 \right) \left( 2n+1 \right) \left( n+1 \right) \left( 2n+1 \right) \left( n+1 \right) \]
\[ + \beta_2 \left( 2n+1 \right) \left( n+1 \right) \left( 2n+1 \right) \left( n+1 \right) \]
\[ + \frac{2n+1}{2} \left( n+1 \right) \left( 2n+1 \right) \left( n+1 \right) \]
\[ + \frac{2n}{2} \left( n+1 \right) \left( 2n+1 \right) \left( n+1 \right) \]
\[ + \frac{2n}{2} \left( n+1 \right) \left( 2n+1 \right) \left( n+1 \right) \]

Residual error of
\[ \theta = 1 + \frac{4}{3} \right) \theta_m + \Pr \left( f_m \theta_m - \frac{2k}{n+1} f_m \theta_m \right) = 0. \]

3. DISCUSSION

The aim of this subsection is to present the effects of pertinent parameters on the velocity, temperature and surface heat transfer. Figs. 3 and 4 are displayed to see the effects of \( \beta_1 \) on the velocity and temperature profiles. It is observed that both the velocity profile and momentum boundary layer thickness decrease by increasing \( \beta_1 \). However, the thermal boundary layer thickness and temperature increase. The dependence of material parameter \( \beta_2 \) on the velocity and temperature profiles are shown in the Figs. 5 and 6 respectively. These Figs. indicate that the velocity profile and momentum boundary layer thickness are increasing functions of \( \beta_2 \) while reverse behavior is observed in the case of temperature and thermal boundary layer thickness.

Fig. 2. \( h \)-curves for the functions \( f \) and \( \theta \).

Fig. 2a. Residual Error for \( f (\eta) \).

Fig. 2b. Residual Error for \( g (\eta) \).

Fig. 3. Influence of \( \beta_1 \) on \( f' (\eta) \).
Variation of parameter \( n \) on the velocity and temperature are sketched in the Figs. 7 and 8. Clearly the effects of \( n \) on the velocity and temperature profiles are quite reverse. Influence of mixed convection parameter \( \lambda \) on both the velocity and temperature profiles are given in the Figs. 9 and 10. It is observed that the velocity and momentum boundary layer thickness increase with the increase of mixed convection parameter \( \lambda \) while the temperature and thermal boundary layer thickness decrease. Figs. 11 and 12 are drawn to see the variation of \( \alpha \) on the velocity and temperature profiles. It is noticed that the velocity and momentum boundary layer thickness increase when \( \alpha \) increases It is also found that the temperature and thermal boundary layer thickness are increasing functions of \( \alpha \). We have drawn Figs. 13 and 14 to see the variation of radiation parameter \( R \) on the velocity \( f'(\eta) \) and temperature \( \theta(\eta) \) profiles.
It is seen that the effect of $R$ on both the temperature and velocity profiles are similar. It is further noted that the momentum and thermal boundary layer thicknesses are increasing functions of $R$. Figs. 15 and 16 are sketched to see the variation of surface temperature parameter $k$ on the velocity $f'(\eta)$ and the temperature $\theta(\eta)$. Both $f'(\eta)$ and $\theta(\eta)$ decrease with the increase in $k$. It is also observed that both the momentum and thermal boundary layer thicknesses decrease when $k$ increases. Influence of Prandtl number $Pr$ on the velocity and temperature profiles are shown in the Figs. 17 and 18. These Figs. show that by increasing the values of $Pr$ both the velocity and temperature profiles decrease.
The momentum and thermal boundary layer thicknesses also decrease by increasing Pr. Clearly an increase in the values of Pr leads to a decrease in the thermal diffusivity. Figs. 19−22 are drawn to see the influence of Deborah numbers $\beta_1$ and $\beta_2$, mixed convection parameter $\lambda$, wedge angle $\alpha$, radiation parameter $R$, Prandtl number $Pr$, surface temperature parameter $k$ and velocity index $n$ on the local Nusselt number $-\theta'(0)$. It is observed from Fig. 19 that $\beta_1$ and $\beta_2$ have quite opposite effects near the wall on the local Nusselt number $-\theta'(0)$ i.e. an increase in $\beta_1$ leads to a decrease in local Nusselt number $-\theta'(0)$. Away from the wall, $-\theta'(0)$ decreases for increasing values of $\beta_2$. Fig. 20 depicts the effects of $\lambda$ and $\alpha$ on $-\theta'(0)$. It is noticed that $-\theta'(0)$ increases through increase of mixed convection parameter $\lambda$ and wedge angle $\alpha$. Fig. 21 depicts that variation of $n$ and $k$ have opposite effects on $-\theta'(0)$. It is noted that $-\theta'(0)$ is decreasing function of $R$ while increasing function of Pr (see Fig. 22). A close look at Table 1 indicates that $25^{th}$-order approximation gives convergent series solutions.

Table 1 Convergence of the homotopy solutions for different order of approximation when $Pr = 1.0$, $\lambda = 0.3$, $n = 1.5$, $k = 0.5$, $R = 0.3$, $k = 0.5$, $\alpha = \pi / 4$, $\beta_1 = 0.2$, $\beta_2 = 0.3$ and $h_f = h_\theta = -0.5$

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>$-f'/0(0)$</th>
<th>$-\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.90330</td>
<td>0.78889</td>
</tr>
<tr>
<td>5</td>
<td>0.86304</td>
<td>0.64042</td>
</tr>
<tr>
<td>10</td>
<td>0.86077</td>
<td>0.62757</td>
</tr>
<tr>
<td>15</td>
<td>0.86047</td>
<td>0.62627</td>
</tr>
<tr>
<td>20</td>
<td>0.86045</td>
<td>0.62616</td>
</tr>
<tr>
<td>25</td>
<td>0.86044</td>
<td>0.62616</td>
</tr>
<tr>
<td>30</td>
<td>0.86044</td>
<td>0.62616</td>
</tr>
</tbody>
</table>

4. CONCLUSIONS

Mixed convection effects in the Falkner-Skan wedge flow of an Oldroyd-B fluid are investigated. Analysis is modeled and analyzed in the presence of thermal radiation. The following points are worth mentioning:

Table 1 shows that convergence of the functions $f$ and $\theta$ are obtained at $25^{th}$-order approximations up to five decimal places when $h_f = h_\theta = -0.5$.

Thermal boundary layer thickness increases with the material parameter $\beta_1$ while reverse behavior is seen in case of momentum boundary layer thickness.

Influence of mixed convection parameter $\lambda$ increases the velocity and momentum boundary layer thickness while it decreases the temperature and thermal boundary layer thickness.

An increase in material parameter $\beta_2$ increases the velocity and reduces the thermal boundary layer thickness.

Both temperature and thermal boundary layer thickness decrease when the Prandtl number $Pr$ is increased.
Influence of wedge angle $\alpha$ and radiation parameter $R$ on both the temperature and velocity profiles are quite similar. Thermo boundary layer and momentum boundary layer thicknesses are decreasing functions of surface temperature exponent $k$.

Surface heat transfer $-\theta(0)$ increases with an increase of wedge angle $\alpha$, mixed convection parameter $\lambda$, radiation parameter $R$, Prandtl number $Pr$ and surface temperature exponent $k$.

In case of material parameter $\beta_1$ the surface heat transfer $-\theta(0)$ decreases while the material parameter $\beta_2$ decreases it near the boundary and increases far away.

**ACKNOWLEDGMENT**

The research work of Dr. Alsulami was partially supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

**REFERENCES**


Kuo, B. L. (2005). Heat transfer analysis for the Falkner-Skan wedge flow by the differential


