Theoretical Analysis of Shear Thinning Hyperbolic Tangent Fluid Model for Blood Flow in Curved Artery with Stenosis

S. Nadeem and S. Ijaz

Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000 Pakistan

†Corresponding Author Email: Shagufta.me2011@yahoo.com

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ABSTRACT

In this paper, we have considered the blood flow in a curved channel with abnormal development of stenosis in an axis-symmetric manner. The constitutive equations for incompressible and steady non-Newtonian tangent hyperbolic fluid have been modeled under the mild stenosis case. A perturbation technique and homotopy perturbation technique have been used to obtain analytical solutions for the wall shear stress, resistance impedance to flow, wall shear stress at the stenosis throat and velocity profile. The obtained results have been discussed for different tapered arteries i.e., diverging tapering, converging tapering, non-tapered arteries with the help of different parameters of interest and found that tapering dominant the curvature of the curved channel.

Keywords: Curved artery; Blood flow; Stenosis; Analytical solutions; Hyperbolic tangent fluid model.

NOMENCLATURE

- \( a \) indicate stenosis position
- \( b \) length of stenosis
- \( C_i \) constants (i=1-10)
- \( e_o \) radius of normal artery
- \( F \) flow rate
- \( h(x) \) height of stenosis
- \( L \) length of stenosed artery
- \( n \geq 2 \) stenosis shape
- \( n_e \) fluid parameter
- \( P \) pressure
- \( Q \) embedding parameter
- \( R_e \) Reynolds number
- \( r \) radial direction
- \( R_c \) curvature
- \( S_{rr}, S_{r\alpha}, S_{\alpha\alpha} \) shear stress
- \( u_o \) averaged velocity
- \( (-,*) \) denotes the non-dimensional quantities
- \( U, V \) components of velocity
- \( w_e \) Weissenberg number
- \( x \) axial direction
- \( \varepsilon \) tapering parameter
- \( \gamma_0 \) extra stress tensor
- \( \delta \) maximum height of stenosis
- \( \lambda \) resistance impedance
- \( \lambda_o \) resistance (no stenosis)
- \( \mu_e \) shear rate viscosity at infinity
- \( \mu_0 \) shear rate viscosity at zero
- \( \tau_o \) shear stress (no stenosis)
- \( \Pi \) second invariant strain tensor
- \( \gamma \) shear rate
- \( \Gamma \) time constant
- \( \varphi \) tapering parameter
- \( \eta, \sigma, \zeta, \varepsilon, a_j (j=1-4) \) constants

1. INTRODUCTION

Most of the cardiovascular diseases, especially atherosclerosis or stenosis have been found to be responsible for major deaths in both developing and developed countries. It occurs to deposition of proliferation in the connective tissues and cholesterol in the arterial wall. The presence of
stenosis at one or more major location may lead to disorders in circulatory systems and different arterial diseases such as myocardial, angina pectoris, coronary thrombosis, infarction and strokes etc. Young et al. (1973) investigated axisymmetric and nonsymmetric plastic models. Hassan et al. (2008) investigated the laminar sinusoidal pulsating flow through a modeled arterial stenosis with a trapezoidal profile. Mandal et al. (2010) investigated numerical solutions of the steady viscous flow due to arterial disease through different double stenosed artery. Nadeem et al. (2015) investigated the viscous fluid model through an axis-symmetric stenosis with the effect of three different types of arteries.

The rheological study of blood flow has several goals such as not only to understanding health and disease issues but also that what kind of fluid it is in nature. Some researchers considered blood as a Newtonian fluid especially flows of blood in the large vessels such as the aorta. Liu et al. (2004) investigated the pulsating blood flow through models of stenotic and tapered arteries to discuss the distributions of the wall shear stress. In fact blood flow through constricted arteries behaves like a non-Newtonian fluid at low shear rates being suspension of plasma, white cells, red cells and platelets. Canic et al. (2003) investigated the quasilinear effects arising in a hyperbolic system of partial differential equations modelling blood flow through large compliant. Sankar et al. (2004) discussed the effects of catheterization and non-Newtonian nature of blood in small arteries mathematically by modeling blood as a Herschel-Bulkley fluid. Mekheimer and ElKot (2008) discussed the micropolar fluid model for axis-symmetric blood flow through radially symmetric but axially nonsymmetric mild stenosis tapered arteries. Sankar et al. (2009) discussed the mathematical model of pulsatile flow of non-Newtonian fluid in stenosed arteries. Here they discussed the Herschel Bulkley fluid and found an analytical solution by using a regular perturbation method. Nadeem et al. (2012) investigated the blood flow through a tapered artery with a stenosis assuming the flow is steady and treated blood as hyperbolic tangent fluid. Reddy et al. (2014) investigated the blood flow between the clogged (stenotic) artery and the catheter with asymmetric nature of the stenosis. Jung et al. (2014) investigated the hemodynamics behavior of the blood flow in the presence of the arterial stenosis. Ellahi et al. (2014) discussed the blood flow model through composite stenosis. Here they treated blood as a micropolar fluid model under the mild stenosis case. Nadeem et al. (2009 and 2015) investigated the two-dimensional equations of tangent hyperbolic fluid using the assumptions of low Reynolds number and long wavelength in the studies that are cited above, the arteries carrying blood were considered to be horizontal. It is widely known that many vessels in physiological systems are not the horizontal because some have the inclination and some are the curve. Chakraborty et al. (2011) discussed the suspension model of blood flow through an inclined tube by considering an axially non-symmetric stenosis. However no work has done yet for blood flow in a curved artery (channel) with stenosis but some work has done for peristalsis. Nadeem et al. (2013) discussed the peristaltic flow of the tangent hyperbolic fluid in a curved symmetric channel with sinusoidal waves.

Motivated by the above analysis we have considered bloodflow in curved arteries having mild stenosis and treated blood as non-Newtonian tangent hyperbolic fluid. The governing equations for a tangent hyperbolic fluid in a curved artery (or curved channel) along with the effects of curvature are modeled and solved analytically with the help of regular perturbation method and homotopy perturbation method. The comparison of both analytical methods shows that the solutions are almost same for small physical parameters. In the end physical phenomena of the present analysis have been discussed by plotting the graph of wall shear stress, resistance impedance to flow by numerical integration, velocity profile and stream lines.

2. FORMULATION OF THE PROBLEM

We consider the flow of an incompressible non-Newtonian tangent hyperbolic fluid in a curved artery of radius $R$ with center $O$, components of velocity in radial $U$ and axial $X$ directions are $\bar{V}$ and $\bar{U}$ respectively. The equations for conservation of mass and momentum can be written as,

$$\frac{\partial}{\partial r} \left( \left( r^2 + R^2 \right) \bar{V} \right) + r^2 R \frac{\partial \bar{U}}{\partial r} = 0, \hspace{1cm} (1)$$

$$\rho \left( \left( r^2 + R^2 \right) \frac{\partial \bar{V}}{\partial r} + \frac{R^*}{r + R^*} \frac{\partial \bar{U}}{\partial x} - \frac{\bar{U}^2}{r + R^*} \right) + \frac{\bar{P}}{r + R^*} + \frac{1}{r + R^*} \frac{\partial}{\partial r} \left( r \bar{U} \right) = \frac{r^*}{r + R^*} \frac{\partial}{\partial x} \left( S \bar{S} \right) + \frac{r^*}{r + R^*} \frac{\bar{U} - \bar{P}}{r + R^*}, \hspace{1cm} (2)$$

$$\rho \left( \left( r^2 + R^2 \right) \frac{\partial \bar{U}}{\partial r} + \frac{R^*}{r + R^*} \frac{\partial \bar{U}}{\partial x} + \frac{\bar{U}^2}{r + R^*} \right) - \frac{R^*}{r + R^*} \frac{\partial \bar{P}}{\partial x} + \frac{1}{r + R^*} \frac{\partial}{\partial r} \left( r \bar{U} \right) \left( r + R^* \right) \frac{\partial}{\partial x} \left( S \bar{S} \right) + \frac{R^*}{r + R^*} \frac{\partial}{\partial x} \left( S \bar{S} \right), \hspace{1cm} (3)$$

Where the constitutive equation for hyperbolic tangent fluid is given by Nadeem et al. (2009, 2013, 2015)

$$\tau = -\bar{P}A + \bar{S}, \hspace{1cm} (4)$$

$$\bar{S} = \left( \mu_0 + (\mu_{sc} + \mu_0) \tanh(\bar{S}^0) \right) \left( \bar{S}^0 \right), \hspace{1cm} (5)$$
In above equations \( P \) is defined as pressure, \( \mu_\infty \) as shear rate viscosity at infinity, \( \mu_0 \) as shear rate viscosity at zero, \( \Gamma \) as time constant, \( n_e \) as power law index and \( \dot{\gamma} \) as shear rate,

\[
\bar{\gamma} = \sqrt{\frac{1}{2} \left( \sum_{j} \gamma_{j} \sqrt{\gamma_j \mu_j} \right)^2} = \sqrt{\frac{1}{2} \Pi},
\]

(6)

Where \( \Pi = tr(\text{grad}(\bar{\gamma})) + \text{grad}^2 \bar{\gamma} \), which is defined as second invariant strain tensor. Now for the cases \( \Gamma \bar{\gamma} \ll 1 \) and \( \mu_0 = 0 \), Eq. (5) is reduced to Eq. (7) as,

\[
S = (\mu_0(1 + \Gamma \bar{\gamma}^n) \gamma_0^0 = \mu_0(1 + n_e(\Gamma \bar{\gamma} - 1)) \gamma_0^0.
\]

(7)

Where \( \gamma_0^0 = L + L' \) and extra stress tensor for hyperbolic tangent fluid can written as

\[
S_{\pi} = 2 \mu_0(1 + n_e(\Gamma \bar{\gamma} - 1)) \frac{\partial \bar{\gamma}}{\partial r},
\]

\[
S_{\pi r} = S_{\pi \gamma} = \mu_0(1 + n_e(\Gamma \bar{\gamma} - 1)) \frac{(c_r \bar{U} + R^* \bar{V})}{r + R^*} - \frac{\bar{U}}{r + R^*},
\]

\[
S_{\pi x} = 2 \mu_0(1 + n_e(\Gamma \bar{\gamma} - 1)) \frac{(c_r \bar{U} + R^* \bar{V})}{r + R^*} + \frac{\bar{U}}{r + R^*},
\]

(8)

The dimensionless boundary conditions are defined as

\[
\frac{\partial U}{\partial r} = 0 \quad \text{at} \quad r = 0, \quad U = 0 \quad \text{at} \quad r = h.
\]

(18)

The geometry of stenosis in dimensionless form is defined as

\[
h(x) = e(x) [1 - \eta^* (b^{n-1} (x - a) - (x - a)^n)],
\]

\[
a \leq x \leq a + b,
\]

\[
e(x), \quad \text{or} \quad \text{else}
\]

(9)

with

\[
e(x) = e_0 + e x,
\]

(10)

where, in above \( e(x) \) is the radius of stenotic arterial segment, \( e_0 \) is the radius of a non-stenotic arterial segment, \( e \) is tapering parameter, \( b \) is the length of stenosis, where \( a \) indicates its position and \( n \geq 2 \) determine the shape of stenosis. The parameter \( \eta^* \) is given as

\[
\eta^* = \frac{\delta^* n^*}{e_0 b^*(n-1)},
\]

(11)

where \( \delta^* \) is the maximum height of stenosis which is located at

\[
\bar{x} = a + \frac{b}{n^*}.
\]

(12)

Introducing the following non dimensional variables,
3. SOLUTION OF THE PROBLEM

3.1 Regular Perturbation Solution

According to method we may expand velocity and flow rate by considering \( w_e \) as a small parameter

\[
U = U_0 + w_e U_1 + w_e^2 U_2 + ..., \tag{21}
\]

\[
F = F_0 + w_e F_1 + w_e^2 F_2 + ..., \nonumber
\]

With the help of Eq. (21) the solution of Eq. (17) subject to boundary condition (18) takes the following form

\[
U = \frac{R_e (r + R_e)}{1 - n_e} dP(r + R_e) \ln(r + R_e) = + (C_1 + w_e C_3)(r + R_e) + \frac{1}{r + R_e}(C_2 + w_e C_4) + \frac{a^2 w_e}{8(r + R_e)} + \frac{a^2 w_e}{2} (r + R_e) \ln(r + R_e) \tag{22}
\]

We have defined flow rate as

\[
F = \int_0^h UdR, \tag{23}
\]

Substituting in Eq. (22) into Eq. (23) we get pressure gradient as follow

\[
\frac{dP}{dx} = \frac{F - w_e C_6}{C_5}. \tag{24}
\]

Pressure drop \( \Delta P = P \text{ at } x = 0 \) and \( \Delta P = -P \text{ and } x = L \) through the stenosis between the region \( x = 0 \) and \( x = L \) computed from above Eq. (24) can be written as,

\[
\Delta P = \int_0^L \left( \frac{-dP}{dx} \right) dx. \tag{25}
\]

4. IMPEDANCE RESISTANCE

Using Eq. (25), the resistance can be defined as

\[
\tau = \Delta P \frac{F}{L} = \int_0^a K(x) \bigg|_{b=1} dx + \int_a^{a+b} K(x) dx + \int_0^L \left( \frac{K(x)}{1 - n_e} dx \right) \tag{26}
\]

where

\[
K(x) = -\frac{F - w_e C_6}{C_5 F}. \tag{27}
\]

Using Eq. (27) into Eq. (26), takes the form

\[
\tau = \left( \frac{F - w_e C_6}{C_5 F} \right) \left( L - b \right) + \int_a^{a+b} K(x) dx. \tag{28}
\]

5. WALL SHEAR STRESS EXPRESSIONS

After using dimensionless variables wall shear stress can be defined as

\[
\tau_{rx} = \left( 1 - n_e \right) \left( \frac{\partial U}{\partial r} - \frac{U}{r + R_e} \right) + w_e n_e \left( \frac{\partial U}{\partial r} - \frac{U}{r + R_e} \right)^2 \bigg|_{b=h}, \tag{29}
\]

The maximum shear stress at the throat of stenosis is located at \( x = \frac{a}{b} + \frac{1}{a^2} \) and given as©

\[
\tau_{mx} = \tau_{rx} \bigg|_{h=1-\delta}. \tag{30}
\]

Finally the expression for \( \lambda \) and \( S_{rn} \) can be defined as follows:

\[
\lambda = \left( 1 - \frac{b}{L} \right) \left( \frac{F - w_e C_6}{C_5 F} \right) \bigg|_{b=1} + \frac{a+b}{L} \int_a^{a+b} K(x) dx \}, \tag{31}
\]

in which

\[
\lambda = \frac{\lambda}{\tau_o}, S_{rn} = \frac{\tau_{rn}}{\tau_o}, \tau_{rx} = \frac{\tau_{rx}}{\tau_o}, \lambda_o = L, \tau_o = F. \tag{33}
\]

6. HOMOTOPY PERTURBATION SOLUTIONS

In this section solution have been computed by using homotopy perturbation method suggested by He in (2005), we can write Eq. (17) in operator form

\[
0 = L_U + \frac{2w_e n_e}{1 - n_e} \tag{33}
\]
where linear operator and initial guess is defined as

\[ L_U = (r + R_e) \frac{\partial^2 U}{\partial r^2} + (r + R_e) \frac{\partial U}{\partial r} - \frac{\partial^2 U}{\partial r^2} - (r + R_e) \frac{\partial U}{\partial r} \frac{\partial U}{\partial r} \]

\[ U_{10} = \frac{R_e (r + R_e)}{2(1 - n_e)} \frac{\partial P}{\partial r} (r + R_e) \ln (r + R_e) + \frac{C_7 (r + R_e)}{r + R_e} \]

\[ C_7 = C_8 \]

The homotopy perturbation method suggests that we can write Eq. (34) as follow

\[ H(U, Q) = (1 - Q) \left[ L |U| - L |U_{10}| \right] + Q |L |U| + \frac{2w_e n_e}{1 - n_e} \left[ (r + R_e)^2 \frac{\partial^2 U}{\partial r^2} - (r + R_e) \frac{\partial U}{\partial r} \right] \]

\[ \left( \frac{\partial^2 U}{\partial r^2} - \frac{R_e (r + R_e)}{1 - n_e} \frac{\partial P}{\partial r} (r + R_e) \ln (r + R_e) \right) \]

(37)

According to method procedure, we define as

\[ U = U_0 + QU_1 + Q^2 U_2 + \ldots \]

(38)

\[ F = F_0 + QF_1 + Q^2 F_2 + \ldots \]

Where \( Q \) is the embedding parameter, substituting above Eq. (38) into Eq. (37) and take \( Q \rightarrow 1 \), we arrive at

\[ U = \frac{R_e (r + R_e) dP}{2(1 - n_e)} \frac{(r + R_e) \ln (r + R_e)}{(r + R_e) + (C_7 + C_8)} + \frac{C_7}{r + R_e} \]

\[ \left( \frac{\partial^2 U}{\partial r^2} - \frac{R_e (r + R_e)}{1 - n_e} \frac{\partial P}{\partial r} \right) \ln (r + R_e) \]

(39)

Using Eq. (39) in Eq. (23), we get

\[ \frac{dP}{dx} = \frac{F - C_{11}}{C_5} \]

(40)

7. RESULTS AND DISCUSSION

To analyze the imperative aspect in this paper we have considered three different types of tapering effects on a curved arteries having mild stenosis by plotting the graphs of wall shear stress, wall shear stress at the throat of stenosis and impedance resistance to blood flow with the help of the different emerging flow parameters such as stenosis height, stenosis shape, curvature, Weissenberg number and shear thinning fluid parameter by keeping parameters constant as \( F = 0.09, \sigma = 0.01, \delta = 0.09, n_e = 0.1, w_e = 0.1, L = 2, R_e = 3.1, n = 2 \). Figs. (2) to (6) are plotted to show the influence of stresses on the wall of curved arteries in the presence stenosis with axial distance \( x \). It is observed from these figures that stresses on the wall of stenosed arteries start steeply decreasing towards the downstream of the stenotic segment and then start rapidly increasing towards the end of the stenotic segment. The wall shear stress for different values of fluid parameter \( n_e \) is given in Fig. (2). It is observed from this figure that stresses on the wall of curved arteries increases with an increase in shear thinning fluid parameter \( n_e \). Fig. (3) is plotted for different values of Weissenberg number \( w_e \) which is the ratio of the relaxation time and a specific process time of fluid. It is observed from graph that by increasing Weissenberg number there will be increase in relaxation time and when we relax time flow can move easily, so stresses on the wall of curved arteries decreases. The wall shear stress for different values of stenosis height \( \delta \) and curvature \( R_e \) are given in Figs. (4) and (5). It is observed that the stresses on the wall arteries are inversely related to the stenosis height and directly related to the curvature of the curved arteries. The effect of stenosis shape \( n \) with different tapering is given in Fig. (6). It is observed that a stress on the wall of the curved arteries increases between the region \( 0 \leq x \leq 0.57 \), while opposite trend is observed in the rest of region. The velocity profile for different tapering effect is given in Figs. (8) and (9). It is observed from these graphs that the velocity profile increases at the center of the stenosed arteries with an increase in the values of stenosis height \( \delta \) and Weissenberg number \( w_e \) between the interval \( -0.46 \leq r \leq 0.54 \), while opposite trend is observed near the wall of stenosed arteries between the interval \( -0.88 \leq r \leq -0.45 \) and \( 0.55 \leq r \leq 0.89 \). It is observed from these graphs that amplitude for converging tapering is higher at the center of the arteries as compared to the near of the wall of curved arteries. The distribution for wall shear stress at the stenosis throat against the maximum height of stenosis \( \delta \) is plotted through Figs.(10)-(12). The wall shear stress at the stenosis throat for different values of shear thinning fluid parameter \( n_e \) and Weissenberg number \( w_e \) are given in Figs. (10) and (11). It is observed that shear stress at the throat of stenosis for curved arteries increases by increasing the values of shear thinning fluid parameter \( n_e \), while opposite behavior at the throat of stenosis is observed for Weissenberg number \( w_e \). Fig. (12) is plotted for shear stress at throat of stenosis for different values of curvature \( R_e \) and observed stresses at throat of stenosis decreases with an increase in the curvature of the curved arteries. The resistance impedance to blood flow along maximum height of stenosis \( \delta \) for different type of stenosed arteries are plotted from Figs. (13)-(16) and observed in these figures that impedance resistance to blood flow is maximum near the peak of stenosis, which gives higher amplitude for.
Figs. 2. Variation of wall shear stress for $n_s = 0.1$.

Figs. 3. Variation of wall shear stress for $n_s = 0.1$.

Figs. 4. Variation of wall shear stress for $R_c = 3.1$.

Figs. 5. Variation of wall shear stress for $\delta = 0.09$.

Figs. 6. Variation of wall shear stress for stenosis shape $n$.

Figs. 7. Comparison of velocity profile between RPM and HPM.

converging tapering as compare to other associated tapering. The effect of stenosis shape $n$ and curvature $R_c$ on resistance impedance to flow is given in Figs. (13) and (14). It is analyzed that impedance resistance to blood flow decreases by increasing the values of the stenosis shape and curvature of the curved arteries. It is important to note here the resistance to blood flow is maximum for the symmetric stenosis case $n=2$. The effect of the Weissenberg number $\textit{ew}$ on resistance impedance to flow is given in Figs. (15). It is observed that resistance impedance to flow increases with an increase in maximum height of stenosis and decreases with an increase in the Weissenberg number $\textit{ew}$. Fig. (16) is plotted for shear thinning fluid parameter $n_s$ and observed that the resistance impedance to blood flow decreases when the fluid is thinner than thicker. Trapping phenomenon has been discussed to show the blood flow pattern in the curved arteries through Figs. (17)-(19). It is observed here that the trapping is now not symmetric about the central line of the
The trapping phenomenon shows that the symmetry destroys due to increase in curvature of the curved arteries that pushed bolus away from the center of the channel.
Figs. 13. Variation of resistance impedance for $R_e = 3.1$.

Figs. 14. Variation of resistance impedance for $n = \frac{3}{2}$.

Figs. 15. Variation of resistance impedance for $n_e = 0.1$.

Figs. 16. Variation of resistance impedance for $n_w = 0.1$.

Fig. 17. Blood flow pattern for $w_e = 0.01$.

Fig. 18. Blood flow pattern for $w_v = 0.02$.

Fig. 18. Blood flow pattern for $n_e = 0.01$.

Fig. 18. Blood flow pattern for $n_e = 0.09$. 
The wall shear stress gives higher amplitude for diverging tapering, while resistance impedance to flow gives higher amplitude for converging tapering.

The wall shear stress and shear stress at the throat of stenosis possess same behavior along maximum height of stenosis with respect to curvature $c_R$, Weissenberg number $\mu_w$ and shear thinning fluid parameter $n_e$.

The wall shear stress shows inverse and resistance impedance to blood flow shows direct relation with stenosis height $\delta$.

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Applied Fluid Mechanics 1, 25-35.


\[ C_{ii} = -\left(4R_1^3(h + R_1)^3(\alpha_1 - \alpha_2n_1) + \frac{dP}{dx} R_1\right) + \alpha_1(1 + n_1)M_1 + 12h^2R_1 + 18h^2R_1^2 + 12hR_1^3 + 4R_1^4)w_0 + 4R_1^2(h + R_1)^2(\alpha_1 - a\epsilon_R + \frac{dP}{dx} R_1)(\ln R_1 - \ln(h + R_1)))((8(1 + n_1)R_1^2(h + R_1)^2)(h^2 + 2hR_1 + 2R_1^2)), \]

\[ C_{iii} = -\left(\frac{dP}{dx} R_1(h(h + R_1))2(1 + n_1)^2(h^2 + 2hR_1 + 2R_1^2)^2(h^2 + 2hR_1 + 4R_1^2) + \alpha_3n_1\left(\frac{dP}{dx} R_1(h^2 + 6h^2R_1 + 28h^2R_1^2 + 72hR_1^3 + 100h^2R_1^3 + 72h^2R_1^2 + 24R_1^2w_0 + 4R_1^2(h + R_1)^2(4(-1 + n_1)^2(h^2 + 2hR_1 + 2R_1^2)^2 + 3n_1\frac{dP}{dx} R_1(3n_1 + 12h^2R_1 + 12h^2R_1^2 + 4R_1^4)w_0 + (2n_1(1 + n_1)^2(h^2 + 2hR_1 + 2R_1^2)^2 + 3n_1\frac{dP}{dx} R_1(3n_1 + 12h^2R_1 + 18h^2R_1^2 + 12hR_1^3 + 4R_1^4)w_0 + n\frac{dP}{dx} R_1(3n_1 + 12h^2R_1 + 18h^2R_1^2 + 12hR_1^3 + 4R_1^4)w_0)(\ln R_1 - \ln(h + R_1))(\ln R_1 - \ln(h + R_1)))((\ln R_1 - \ln(h + R_1)))((16(1 + n_1)(h^2 + 2hR_1 + 2R_1^2)^2). \]