Fully Developed Flow of Fourth Grade Fluid through the Channel with Slip Condition in the Presence of a Magnetic Field

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ABSTRACT

In this paper, to study the incompressible fully developed flow of a non-Newtonian fourth grade fluid in a flat channel under an externally applied magnetic field, an appropriate analysis has been performed considering the slip condition on the walls. The governing equations, Ohm’s law, continuity and momentum for this problem are reduced to a nonlinear ordinary form. The nonlinear equation with robin mixed boundary condition is solved with collocation (CM) and least square (LSM) methods. The effects of parameters such as non-Newtonian, magnetic field and slip parameters on dimensionless velocity profiles will be discussed. In the end, the results could bring us to this conclusion that collocation and least square methods can be used for solving nonlinear differential equations with robin mixed condition.

Keywords: Collocation method; Channel; Fourth grade fluid; Least Square Method.

NOMENCLATURE

- $A$ pressure gradient
- $A_n$ Rivlin-Erickson tensors
- $B$ total magnetic field
- $B_0$ external magnetic field
- $b$ induced magnetic field
- $c_i$ constant of trial function
- $d$ half distant of parallel plates
- $E$ electric field
- $I$ identity tensor
- $H_a$ Hartmann number
- $N_f$ non-Newtonian parameter
- $p$ pressure
- $p(x)$ a function
- $R(x)$ Residual function
- $t$ time
- $u$ velocity
- $\tilde{u}$ approximate function of $u$
- $V$ velocity field
- $w_i$ weighted function
- $a_1, a_2$ material constants
- $\beta_1, \beta_2, \beta_3$ material constants
- $\gamma_1, \gamma_2$ material constants
- $\delta$ delta function
- $\eta$ dimensionless parameter of channel width
- $\lambda$ slip parameter
- $\mu$ viscosity
- $\rho$ density
- $\sigma$ electric conduction
- $\tau$ stress tensor

1. INTRODUCTION

In recent years, the study on non-Newtonian fluids has gained importance and achieved great significance in the industry and technical operations (Siddiqui et al., 2009; Choudhury and Kumar Das, 2014). The classical Navier-Stokes equations not able to describe and explain features of complex rheological fluids such as: food stuffs, shampoo, blood, synovial, paints, micro fluidics and polymer solutions. This kind of fluids is usually known non-Newtonian fluids, which unlike Newtonian fluids, the ratio of shear stress to shear rate is not linear. Many empirical and semi-empirical non-Newtonian models or constitutive equations have been proposed (Islam et al., 2011). Among these, the fluids of differential type (Dunn and Rajagopal, 1995; Truesdell and Noll, 2004) have received considerable attention.

Because of their simplicity and originality, parallel
plates are often used to simulate the actual flow domain conditions in some materials processing applications such as continuous casting, plastic forming and extrusion. Some research have been carried out to analyze the flow of different classes of materials in ducts and channels using various constitutive equations such as inelastic and linear/nonlinear viscoelastic models (Mokarizadeh et al., 2013; Ali et al., 2010; Hayat et al. 2006, Siddiqui et al., 2006; Keimanesh et al., 2011; Mohyuddin, 2005; Ramesh and Devakar, 2015).

Since get exact analytical solution for a nonlinear problem is not easy, we tend to semi-analytical solutions (Ganji and Languri, 2010; Hashemi Kachapi and Ganji, 2011; Nayfeh, 1985: Abbasi et al., 2014). There are some simple and accurate approximation techniques for solving differential equations called the Weighted Residuals Methods (WRMs). Collocation (CM), Galerkin (GM) and Least Square (LSM) are examples of the WRMs.

Actually, LSM and CM are two of the most effective and convenient solutions for both linear and nonlinear equations and don’t require linearization or small perturbation. Motivated by these facts, we used LSM and CM to obtain the solutions of the fully developed steady flow of a fourth grade fluid between two stationary parallel plates with slip conditions at walls.

This paper by this powerful methods deal to solve nonlinear equation with Robin mixed condition. What is more, the results were evaluated and compared, and the numerical analysis validated and properly documented.

2. GENERAL GUIDELINES

Let us consider the fully developed laminar flow of an electrically conducting fourth grade fluid in a channel as shown in Fig.1. The slip boundary conditions are exerted on walls. The uniform magnetic field, \( B_0 \), is imposed along the \( y \)-axis.

The governing equations, continuity, momentum and Ohm’s law for the problem can be written as follows:

\[
\nabla V = 0, \tag{1}
\]

\[
\rho \frac{D V}{D t} = -\nabla p + \text{div} \tau + J \times B \tag{2}
\]

\[
J = \sigma (E + V \times B) \tag{3}
\]

Where \( V \) is the velocity vector, \( \rho \) the constant density, \( \nabla \) the Nabla operator, \( p \) the pressure, \( \tau \) the stress tensor, and \( D/Dt \) denotes the material derivative. The \( \sigma \) and \( J \) denote electrical conductivity and current density respectively and \( B = B_0 + b \) (\( b \) being the induced magnetic field and \( B_0 \) an external magnetic field), is the total magnetic field and \( E \) is the electric field. It is assumed that the magnetic Reynolds Number is small and the induced magnetic field, \( b \), due to the motion of the electrically conducting fluid is negligible. It is also assumed that the electrical conductivity of fluid, \( \sigma \) is constant and the external electric field is zero.

For the present model we take the velocity field of the form:

\[
V = (a(y), 0, 0). \tag{4}
\]

Under these assumptions the last term in Eq. (2), The Lorentz force per unit volume is given by:

\[
J \times B = -\sigma \frac{D^2}{Dt^2} u, \tag{5}
\]

As discussed in (Siddiqui et al., 2009; Islam et al. 2011), the stress tensor \( \tau \) defining a fourth-grade fluid is given by

\[
\tau = \sum_{i=1}^{4} S_i, \tag{6}
\]

where

\[
S_0 = -\rho I, \quad S_1 = \mu A_1, \quad S_2 = \alpha_1 A_2 + \alpha_2 A_1^2, \\
S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (tr(A_1) A_1), \\
S_4 = \gamma_1 A_4 + \gamma_2 (A_2 A_2 + A_1 A_1) + \gamma_3 A_1^2 \\
+ \gamma_4 (A_2 A_1 + A_1 A_2) + \gamma_5 (tr(A_2) A_2) + \gamma_6 (tr(A_1) A_1) \\
+ (\gamma_7 (tr(A_4) + \gamma_8 (A_4)) A_4), \tag{7}
\]

where \( I \) is the identity tensor, \( \mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \) and \( \gamma_8 \) are material constants. The Rivlin-Ericksen tensors \( A_n \) (Siddiqui et al., 2009; Mohyuddin, 2005) are defined by

\[
A_0 = I, \\
A_n = \frac{DA_n-1}{Dt} + A_{n-1} \left( \nabla V \right) + \left( \nabla V \right)^T A_{n-1}, \quad n \geq 1, \tag{8}
\]

which \( t \) is the transpose symbol.

The continuity (Eq. (1)) is satisfied by Eq. (6) and Eq. (2) can be written in component form as:
functions in the

\[ \phi_n(x,y) \] chosen from a linearly independent set. That is:

\[ \mathbf{\phi} = \sum_{i=1}^{n} \phi_i \]

Because the flow is fully developed, the left side of Eq. (9) is only a function of y. If we differentiating both side of Eq. (10) with respect to x and then integrating with respect to y, we can see that the right side of Eq. (9) is only function of x. Thus, Eq. (9) is valid if it's both side equal to a constant. So,

\[ \mu \left( \frac{d^2 u}{dy^2} (y) \right) + 6(\beta_2 + \beta_1) \left( \frac{d}{dy} u(y) \right)^2 + \left( \frac{d^2 u}{dy^2} (y) \right) \]

\[ -\sigma \mu^2 u(y) B^2 \phi = \frac{d}{dx} \phi = A \]

where \( \beta = \beta_1 + \beta_2 \) and A is a constant. Therefore, the problem reduces to solve the second-order nonlinear ordinary differential. Due to symmetry and slip conditions at either of the two plates, there are the following boundary conditions:

\[ \frac{\partial u}{\partial y} \bigg|_{y=0} = 0, \quad \frac{\partial u}{\partial y} \bigg|_{y=\pm \lambda} = -\lambda u(x) \]

By introducing the following non-dimensional parameters:

\[ \eta = \frac{y}{\alpha}, \quad U(\eta) = \frac{u(x)}{\alpha}, \quad N_\gamma = \frac{\Lambda d^2 \beta}{\mu^2}, \quad H_{\alpha} = \frac{\phi}{\sqrt{\nu}} \]

Substituting these functions into Eq. (11) and Eq. (12), rewriting these equations, we finally obtain the following system of nonlinear equations:

\[ \frac{d^2 U}{dy^2} + 6N_\gamma \left( \frac{d U}{dy} \right)^2 + H_{\alpha} \frac{d^2 U}{dy^2} - Ha^2 U - 1 = 0, \]

\[ \frac{d U}{dy} \bigg|_{y=0} = 0, \quad \frac{d U}{dy} \bigg|_{y=1} = -\lambda U(1) \]

2.1 Collocation Method (CM)

Suppose we have a differential operator \( D \) is acted on a function \( u \) to produce a function \( p \) (Hatami et al. 2013):

\[ D(u(x)) = p(x) \]

We wish to approximate \( u \) by a function \( \tilde{u} \), which is a linear combination of basic functions chosen from a linearly independent set. That is:

\[ u \equiv \tilde{u} = \sum_{i=1}^{n} C_i \phi_i \]

Now, by substituting Eq. (17) into the differential operator \( D \), the result of the operations is not \( p(x) \) in general. Hence an error or residual will exist:

\[ E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0 \]

The notion in the collocation is to force the residual to zero in some average sense over the domain. That is:

\[ \int R(x) W_i(x) dx = 0 \quad i = 1, 2, \ldots, n \]

Where the number of weight functions \( W_i \) are identically equal the number of unknown constants \( C_i \) in \( \tilde{u} \). The result is a set of \( n \) algebraic equations for the unknown constants \( C_i \). For collocation method, the weighting functions are taken from the family of Dirac \( \delta \) functions in the domain. That is, \( W_i(x) = \delta(x-x_i) \). The Dirac \( \delta \) function has the property that:

\[ \delta(x-x_i) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{Otherwise} \end{cases} \]

And residual function must force to be zero at specific points.

2.2 Application of CM on Problem

Consider the trial function as:

\[ u(\eta) = \frac{1}{2} c_i \lambda \eta^3 + \frac{1}{3} c_i \lambda \eta^2 + \frac{1}{2} c_i \lambda \eta \]

\[ + \frac{1}{2} c_i \lambda \eta^4 + c_i \left( \frac{1}{2} \lambda - 1 \right) + c_i \left( -\frac{1}{3} \lambda - 1 \right) \]

\[ + c_i \left( -\frac{1}{4} \lambda - 1 \right) + c_i \left( \frac{1}{5} \lambda - 1 \right) \]

The trial function satisfies the boundary condition in Eq. (15) and setting into Eq. (14), residual function,

\[ R(c_i, c_2, c_3, \ldots) \], is found as:

\[ R(\eta) = 1 + 24N_f c_i \lambda \eta^3 c_i + 30N_f c_i \lambda \eta^2 c_i + 30N_f c_i \lambda \eta c_i + 30N_f c_i \lambda \eta^4 c_i + 48N_f c_i \lambda \eta^3 c_i + 60N_f c_i \lambda \eta c_i + 60N_f c_i \lambda \eta^2 c_i + 12N_f c_i \lambda \eta^4 \]

\[ + 6N_f c_i \lambda \eta c_i + 24N_f c_i \lambda \eta^4 c_i + 0.333Ha c_i \lambda \eta^4 c_i + 84N_f c_i \lambda \eta^2 c_i - 0.2Ha c_i \lambda \eta^4 c_i + 0.333Ha c_i \lambda \eta^2 c_i + Ha c_i \lambda c_i + 0.333Ha c_i \lambda \eta^4 c_i + 0.72N_f c_i \lambda \eta^3 c_i + c_i \lambda + 0.5Ha c_i \lambda + 0.333Ha c_i \lambda \]

\[ + 2c_i \lambda \eta^2 + 0.2Ha c_i \lambda + 3c_i \lambda \eta^4 - 0.5Ha c_i \lambda \eta^4 c_i + 36N_f c_i \lambda \eta^2 c_i + 0.25Ha c_i \lambda \eta^4 + 108N_f c_i \lambda \eta^3 c_i + 42N_f c_i \lambda \eta c_i + 4c_i \lambda \eta = 0 \]
On the other hands, the residual function must be close to zero. For reaching this importance, for specific points in the domain \( \eta \in [0,1] \) should be chosen. These points are selected as:

\[
\eta_{12} = 0, \quad \eta_{34} = 0, \quad \eta_{23} = 0, \quad \eta_{21} = 0
\]

(23)

For example the first equation is written as:

\[
23.25 \lambda \frac{\partial c_i}{\partial \eta} + 12 \frac{\partial c_i}{\partial \eta} = 0
\]

(24)

The rest of equations are written similarly. Finally by substitutions the \( \lambda, N_f, \) and \( H_a \) into the residual function, \( R(c_0, c_1, c_2, c_3, \eta) \), a set of four equations and four unknown coefficients are obtained. After solving these equations for unknown parameters \((c_0, c_1, c_2, c_3)\), the velocity distribution equation will be determined that shows in table 1.

### 3.1. Principles of Least Square Method

If the continuous summation of all the squared residuals is minimized, the rationale behind the name can be seen. In other words:

\[
S = \int R(x) R(x) dx = \int R^2(x) dx
\]

(25)

In order to achieve a minimum of this scalar function, the derivatives of \( S \) with respect to all the unknown parameters must be zero. That is,

\[
\frac{\partial S}{\partial c_i} = 2 \int R(x) \frac{\partial R}{\partial c_i} dx = 0
\]

(26)

Comparing with Eq. (19), the weight functions are seen to be

\[
W_i = \frac{\partial R}{\partial c_i}
\]

(27)

However, the "2" coefficient dropped, since it cancels out in the equation. Therefore the weight functions for the least squares method are just the derivatives of the residual with respect to the unknown constants.

### 3.2. Application

Consider the trial function and corresponding residual as Eq. (21) and Eq. (22) and using Eq. (27), the first weight function is obtained as,

\[
W_i = \frac{\partial R}{\partial c_i}
\]

(28)

In this study, we employed CM and LSM to find the velocity field for fully developed steady flow of a fourth grade fluid between two stationary parallel plates under transverse magnetic filed. The solutions are shown graphically, because they were too long to be mentioned here.

The comparison of results between the applied methods, CM and LSM and Numerical Methods, for different values of active parameters is shown in Figure. 2. The numerical solution is performed using the algebra package Maple 16.0, to solve the present case. The package uses a fourth order Runge–Kutta procedure for solving nonlinear boundary value (B-V) problem (Hatami et al. 2013; Hatami and Ganji 2014). Validity of LSM is shown in Table 2. In these tables, the Error is defined as:

\[
\text{Error} = \frac{\text{LSM} - \text{CM}}{\text{LSM}}
\]
The results show that the solution methods are precise and this investigation is completed by depicting the effects of some important parameters by Least Square Method to evaluate how these parameters influence on this fluid.

As shown, an increase in the magnetic parameter leads to decrease in the velocity components at given point as can be seen from Fig. 3. This is due to the fact that applied transverse magnetic field produces a drag in the form of Lorentz force thereby decreasing the magnitude of velocity. In addition, Fig. 4 shows the effect of non-Newtonian parameter $N_f$ on the velocity components for $\lambda = 0.5$, $Ha = 1$. It is noticed that an increase in dimensionless parameters $N_f$ tends to decrease the velocity profile $U(\eta)$. It is worth mention that, the same effect is observed for the slip parameter which is depicted by the Fig. 5. This is due to the fact that, with the increasing of slip parameter some part of fluid molecules strike solid surface and reflected diffusely increases then velocity decreases.

Table 2 The results of CM, LSM and Numerical methods for $U(\eta)$ for $\lambda = 0.4$, $Ha = 1$ and $N_f = 1$

<table>
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4. CONCLUSION

In this study, a fully developed steady flow of a fourth grade fluid between two stationary parallel plates was analyzed using Least Square Method (LSM) and Collocation Method (CM). LSM and CM does not require small parameters in the equation so that the limitations of the traditional perturbation methods can be eliminated and thereby the calculations are straightforward. Effects of different physical parameters such as Slip number, the Non-Newtonian number and the magnetic field parameter on the velocity profiles of the problem have been investigated. As an important outcome from the present study, it can be observed that the results of LSM are more accurate than CM and they are in excellent agreement with numerical ones, so LSM can be used for finding analytical solutions of non-Newtonian problems easily. Also it can be concluded that increasing the magnetic parameter...
leads to decrease in velocity values in whole domain. In addition, increasing in slip parameter caused a decrease in velocity components. It was shown that proposed methods provide simple, accurate and appropriate techniques for solving nonlinear differential equations with Robin mixed.

Fig. 4. Dimensionless velocities predicted by LSM in different $N_f$ number when $\lambda = 0.5, Ha = 1$.

Fig. 5. Dimensionless velocities predicted by LSM in different $\lambda$ number when $N_f = 0.5, Ha = 1.5$.

REFERENCES


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## Table 3 The results of CM, LSM and Numerical methods for $U(\eta)$ for $\lambda=0.4, H_0=1$ and $N_f = 1$

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