



Variational Approach of Marangoni Mixed Convection Boundary Layer Flow with Pade Technique

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(Received September 9, 2015; accepted February 1, 2016)

ABSTRACT

In this paper, Variational Iteration method with combining Pade approximation (Modified Variational Iteration Method-MVIM) is performed to Marangoni convection flow over the surface with buoyancy effects which is occurred gravity and external pressure. After the appropriate transformation of equations, we get the dimensionless form to solve numerically with modified variational iteration method. We compare the our results with well-known asymptotic expansion method used by Zhang Yan and Zheng Liancun and also compare with Fourth order Runge Kutta solution which are presented in tables. Very efficient and accurate results are obtained with presented method.

Keywords: Variational iteration method; Marangoni convection; Pade approximation; Boundary layer flow.

NOMENCLATURE

f	velocity function	y	Cartesian coordinates normal to S
g	gravitational acceleration	x	Cartesian coordinate along
Pr	Prandtl number		
S	interface	λ	Maranon mixed convection parameter
T	temperature	η	similarity variable
ΔT	positive Temp. increment along the interface	Γ	buoyancy forces parameter
x	component of velocity	σ	surface tension
$u_e(x)$	velocity outside boundary layer	θ	temperature function
y	component of velocity	φ	Lagrange multiplier
		Ψ	stream function

1. INTRODUCTION

Marangoni convection flow is stimulated by variations of surface tension throughout liquid-liquid or liquid-gas surfaces and it is important in many fields of nature and engineering. Fundamental treatment of Marangoni flow has been analyzed by (Gelles 1978; Napolitano 1982, 1979, 1978; Okano *et al.* 1989). Some of numerical studies depend on the Marangoni convection in various geometries have been presented by (Golia and Viviani 1985, 1986), (Cristopher and Wang 2001), (Pop *et al.* 2001), (Chamkha *et al.* 2006), (Arafune and Hirata 1999), (Magyari and Chamkha 2008), (Aini *et al.* 2012), (Hamid and Arifin 2014), (Remeli *et al.* 2013), (Zhang and Zheng 2012), (Chen 2007) and (Al-Mudhaf and Chamkha 2005).

In real applications, mathematical problems are usually modeled by nonlinear ordinary and partial differential equations such as physical and engineering applications. In generally nonlinear models may not be an exact solutions. Therefore, we try to find approximate or numerical solutions of these models as seen in references (Freidoonimehr and Rashidi 2015; Jhankal 2015). One of the most popular technique is variational iteration method that is very powerful.

In this context, we will consider to extend the work studied by (Zhang and Zheng 2014) in order to find analytical solutions by using variational iteration method with combining Pade approximation called modified variational iteration method (MVIM). Also, we will give comparison between MVIM and

asymptotic expansion method (Zhang and Zheng 2014). Additionally, unknown parameters of velocity gradient $f'(0) = a$ and temperature gradient $\theta'(0) = b$ which obtained by MVIM is compared with Runge-Kutta method as shown in tables. The effects and variations of Pr, λ and Γ on velocity and temperature profiles are presented graphically. All the calculations for solutions are provided by only one or two step iterations. Thus, it is found that the present results are in very good agreement with other known result as presented in Table 1, 2 and 3.

2. MATHEMATICAL DESCRIPTION OF PROBLEMS

We will consider two-dimensional Marangoni boundary layer flow with buoyancy effects due to external pressure gradient and gravity. It's occurs along an interface S between two fluids as in Fig. 1.

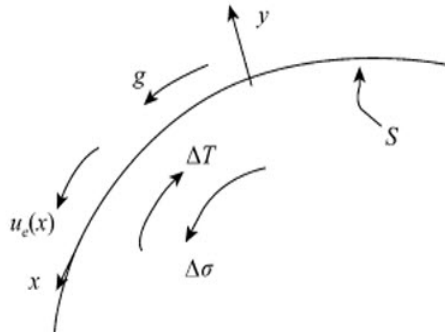


Fig. 1. Schematic of the problem.

Gravity g occurs throughout on interface S , the surface tension changes with temperature. Viscous dissipation and interface tortuosity are negligible. Also the flow fields for two interfacing fluids are independent (Golia and Viviani 1985, 1986; Zhang and Zheng 2014).

Considering these information, we can write the governing equations for the Marangoni boundary layer with water based fluid as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \frac{\partial^2 u}{\partial y^2} - \Gamma \lambda T \quad (1)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2}$$

Also boundary conditions are given as

$$\text{if } y = 0, \frac{\partial u}{\partial y} = -\frac{\partial T}{\partial x}, v = 0, T = T_0 x^5$$

$$\text{if } y \rightarrow \infty, u \rightarrow u_e(x) = u_0 x^3, T \rightarrow 0 \quad (2)$$

where u and v are the velocity components corresponding to x and y axis respectively. $u_e(x)$ is external velocity, T is fluid temperature, λ is Marangoni mixed convection parameter. Also, if $\Gamma = -1$, then buoyancy force is available and if $\Gamma = 1$, then buoyancy force is not available. Additionally, variation of surface tension as

$$\sigma_T = -\frac{d\sigma}{dT}, \sigma = \sigma_m - \sigma_T(T - T_m). \quad (3)$$

By using these facts and boundary conditions, we can write the transform variables as

$$\eta = \sqrt[4]{T_0} xy, T = -T_0 x^5 \theta(\eta)$$

$$\Psi(x, y) = \sqrt[4]{T_0} x^2 f(\eta) \quad (4)$$

$$\frac{u_0}{\sqrt{T_0}} = r, \sqrt[4]{T_0} = k$$

Combining (1)-(4) and considering the literature (Golia and Viviani 1985, 1986; Chamkha *et al* 2006; Al-Mudhaf and Chamkha 2005) we obtain the main ordinary differential system which is the reduced form of (1) as

$$f''' - 3(f')^2 + 2ff'' + 3r^2 + \Gamma \lambda \theta = 0 \quad (5)$$

$$\theta'' + Pr(2f\theta' - 5f'\theta) = 0$$

and boundary conditions turn into

$$f(0) = 0, f'(\infty) = r = 1,$$

$$f''(0) = -5k = -1, \quad (6)$$

$$\theta(0) = 1, \theta(\infty) = 0$$

Here, prime denotes the derivatives with respect to η .

3. MODIFIED VARIATIONAL ITERATION METHOD (MVIM)

Variational iteration method (VIM) is one of the powerful mathematical tool to solve various kinds of linear and nonlinear problems as shown some of in ref. (He 2007, 1999, 1997; He and Hong 2007). In order to basic definition of VIM, we consider the following general nonlinear problem (He 2007, 1999, 1997; He and Hong 2007)

$$L[u(x)] + R[u(x)] + N[u(x)] = g(x) \quad (7)$$

where $L = \frac{d^m}{dx^m}$, $m \in \mathbb{N}$ and R are linear operators,

N is a nonlinear operator and g is given continuous function. According to the originally VIM, we construct the correction functional as

$$f_0(\eta) = a\eta - \frac{\eta^2}{2} \text{ and} \quad (17)$$

$$\theta_0(\eta) = b\eta + 1$$

where a, b are unknown coefficients that will be obtained by applying boundary conditions (6). By using the variations theory (9)-(10), Lagrange multipliers are found as follow respectively

$$\varphi_f = -\frac{1}{2}(s-\eta)^2 \text{ and } \varphi_\theta = (s-\eta) \quad (18)$$

Thus, Lagrange multipliers (18) put into (8), then successive iteration equations are written as

$$f_{n+1}(\eta) = f_n(\eta) - \frac{1}{2} \int_0^\eta (s-\eta)^2 \left(\begin{array}{l} f_n'''(s) - 3(f_n'(s))^2 \\ + 2f_n(s)f_n''(s) + 3r^2 \\ + \Gamma\lambda\theta_n(s) \end{array} \right) ds \quad (19)$$

and

$$\theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^\eta (s-\eta) \left(\begin{array}{l} \theta_n''(s) \\ + \text{Pr} \left(\begin{array}{l} 2f_n(s)\theta_n'(s) \\ - 5f_n'(s)\theta_n(s) \end{array} \right) \end{array} \right) ds \quad (20)$$

By applying (17) to (19)-(20), we obtain the solutions of (5) respect to boundary conditions (6) as follow

$$f(\eta) = a\eta - \frac{1}{2}\eta^2 + \left(\begin{array}{l} -\frac{1}{6}\Gamma\lambda \\ + \frac{1}{2}a^2 - \frac{1}{2} \end{array} \right) \eta^3 + \left(\begin{array}{l} \frac{1}{6}a - \frac{1}{24}\Gamma\lambda b \end{array} \right) \eta^4 + \left(\begin{array}{l} \frac{\Gamma\lambda\text{Pr}a}{24} + \frac{a^3 - a}{20} + \frac{2 - \Gamma\lambda a}{60} \end{array} \right) \eta^5 + \left(\begin{array}{l} \frac{\Gamma\lambda\text{Pr} + 2\Gamma\lambda + 1 - a^2}{144} + \frac{1 - a^2}{24} \end{array} \right) \eta^6 + \dots \quad (21)$$

$$\theta(\eta) = 1 + b\eta + \frac{5}{2}\text{Pr}a\eta^2 + \left(\begin{array}{l} -\frac{5\text{Pr}}{6} + \frac{ab\text{Pr}}{2} \end{array} \right) \eta^3 + \left(\begin{array}{l} \frac{5(a^2\text{Pr}^2 - \Gamma\lambda\text{Pr})}{24} \\ + \frac{5\text{Pr}(a^2 - 1) - \text{Pr}b}{8} \end{array} \right) \eta^4 \quad (22)$$

$$+ \left(\begin{array}{l} \frac{13(\text{Pr}a^2b - \text{Pr}b) - \text{Pr}^2a^2b}{40} \\ + \frac{3\Gamma\lambda\text{Pr}b}{20} - \frac{\text{Pr}a + 2\text{Pr}^2a}{6} \end{array} \right) \eta^5 + \left(\begin{array}{l} \frac{\text{Pr} + 2\text{Pr}^2}{36} - \frac{11\Gamma\lambda\text{Pr}^2a}{72} \\ + \frac{4a\text{Pr}b + \text{Pr}b^2\Gamma\lambda}{40} \\ + \frac{11\text{Pr}^2a^3 - \text{Pr}^2a}{24} \end{array} \right) \eta^6 + \dots$$

For (21) and (22), iteration process continues sufficiently (as seen (11)).

In the series solutions (21) and (22), the unknown constant a, b which are denoted to velocity gradient and temperature gradient respectively, are found by applying very efficient approximation called Pade Technique (12-16) and infinite boundary conditions (6).

For numerical values of $\lambda=1$, $\text{Pr}=1$ and $\Gamma=+1$ (opposing buoyancy force), we find unknown constants as

$$f'(0) = a = 1.412274065 \quad (23)$$

$$\theta'(0) = b = -2.803862335$$

From (23) and put $\lambda=1$, $\text{Pr}=1$ and $\Gamma=+1$ into (21-22), then numerical solution of (5) with MVIM is found as follow respectively,

$$f(\eta) = 1.412274065\eta + 0.3305923508\eta^3 - 0.1185514135\eta^4 + 0.02117728081\eta^5 - 0.00410565956\eta^6 - \frac{1}{2}\eta^2 \pm \dots$$

$$\theta(\eta) = 1 - 2.803862335\eta + 3.530685162\eta^2 - 2.813244362\eta^3 + 1.763385808\eta^4 - 1.052008622\eta^5 + 0.7107541919\eta^6 \pm \dots$$

5. RESULTS And DISCUSSION

The dimensionless form (5) and (6) of Marangoni boundary layer flow equations (1) and (2) are considered. These equations have been solved by modified variational iteration method (MVIM) and very efficient and accurate results are obtained by MVIM.

For both opposing and favorable buoyancy effected Marangoni flow with various values of Prandtl number (Pr) and Marangoni mixed convection parameter λ , our results are compared with well known asymptotic iteration method (Zhang and Zheng 2014) and fourth order Runge-Kutta method in Table 1,2,3.

Table 1 Comparison of velocity and temperature gradients values for buoyancy force effects

$\Gamma = +1$		MVIM		Ref (Zhang & Zheng)		Numerical ¹	
λ	Pr	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$
1	1	1.412274065	-2.803862335	1.3770953	-2.9458683	1.4113699382952547	-2.78948714536900289
1	2	1.405165167	-3.976787667	1.3598498	-4.2130978	1.4002194673439794	-3.97480693986123512
0	0.5	1.332167618	-1.880343197	1.3287628	-2.0611599	1.3531950142211835	-1.90716896300225147
2	0.5	1.482416887	-2.056431944	1.4573970	-2.0884821	1.4889922387434242	-1.99825843240349954
$\Gamma = -1$		MVIM		Ref (Zhang & Zheng)		Numerical ¹	
λ	Pr	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$
1	1	1.287203220	-2.640556677	1.2768744	-2.8990230	1.2917523393322974	-2.67069088539676658
1	2	1.315836672	-3.940255478	1.2981999	4.1678960	1.3040071102798589	-3.83725279754441573
1	3	1.334733660	-4.946965767	1.3153168	-7.9219190	1.3105560074629494	-4.7380888016884386
2	1	1.211855517	-2.528374125	1.2038062	-2.8498573	1.2264015411187497	-2.60354476159533110
3	1	1.122504874	-2.387794592	1.1230572	-2.8120538	1.1562579654494212	-2.52953330807238563

¹Fourth order Runge Kutta

Table 2 Comparison of Temperature and Velocity gradients for various values of Pr and λ for $\Gamma = +1$

$\Gamma = +1$		MVIM		Numerical ¹	
λ	Pr	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$
0.5	1	1.385624006	-2.739988206	1.3826578393120732	-2.76143002511236446
0.5	1.5	1.381628631	-3.392163066	1.37926876062386006	-3.40135895617698658
0.5	2	1.378716936	-3.925101657	1.37695951454900345	-3.94198486617161148
0	1	1.356249272	-2.649314461	1.35319501463357117	-2.7323451676649424
1.5	1	1.441773170	-2.858328650	1.43938712832371896	-2.8166003984916661
2	1	1.426551360	-2.829817768	1.46675860211244036	-2.84284309058585061

¹Fourth order Runge Kutta

Table 3 Comparison of Temperature and Velocity gradients for various values of Pr and λ for $\Gamma = -1$

$\Gamma = -1$		MVIM		Numerical ¹	
λ	Pr	$f'(0)$	$\theta'(0)$	$f'(0)$	$\theta'(0)$
0.5	1	1.319306864	-2.686759875	1.32291768517141661	-2.70213587974090652
0.5	1.25	1.324084466	-3.053502073	1.32489599762663102	-3.03489176113386927
0.5	1.5	1.328280246	-3.385784692	1.32647661498123703	-3.3364729195401881
0	1	1.346509763	-2.72521503	1.35319502837390648	-2.73234497964024126
1.5	1	1.251247832	-2.587702322	1.25961368372507465	-2.63787843974779923
2	1	1.211855517	-2.528374125	1.22640155716212185	-2.60354451673469889

¹Fourth order Runge Kutta

It is evident from Table 1 that our results better than the results in ref. (Zhang and Zheng 2014) with compared numerical method. Also from Table 2 and 3, it tells us that our presented method (MVIM) is efficient and powerful mathematical tool.

Fig. 2-3 demonstrate the variations of velocity profiles for both opposing $\Gamma=+1$ and favorable $\Gamma=-1$ buoyancy forces with different values of Prandtl Number (Pr).

Also, Fig. 4-7 show the variations of temperature profiles as the same meaning of Fig. 2-3 with various values of Pr and Marangoni mixed

convection parameter λ respectively.

6. CONCLUSION

In this paper, we consider the nonlinear ordinary differential equations which corresponds to Marangoni boundary layer flow with buoyancy effects. These equations are solved by Modified variational iteration method analytically. The velocity gradient $f'(0)$, temperature gradient $\theta'(0)$ as well as the temperature and velocity profiles are analyzed and compared for buoyancy

opposed and buoyancy favorable cases. Consequently, results show that MVIM is very powerful and convenient method for analytical and numerical solutions for nonlinear flow equations.

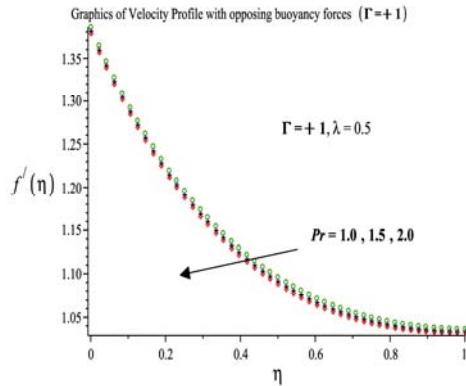


Fig. 2. Prandtl Number effects on velocity profile for eqn. (5) with opposing buoyancy forces.

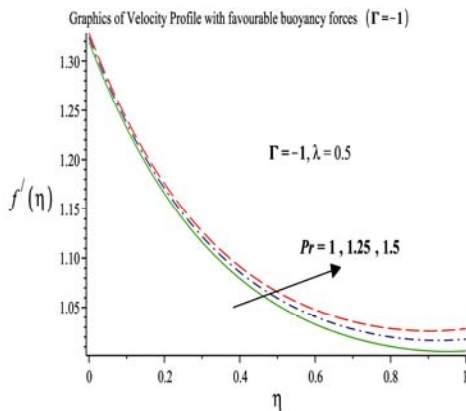


Fig. 3. Prandtl Number effects on velocity profile for eqn. (5) with assisting buoyancy forces

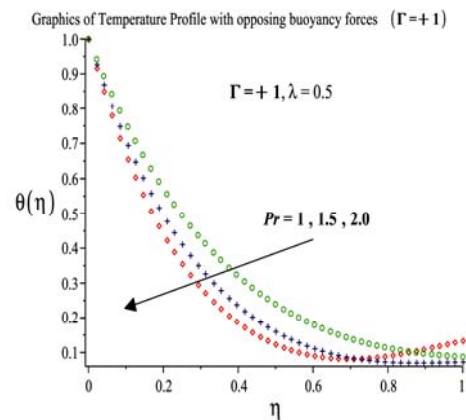


Fig. 4. Prandtl Number effects on temperature profile for eqn. (5) with opposing buoyancy forces.

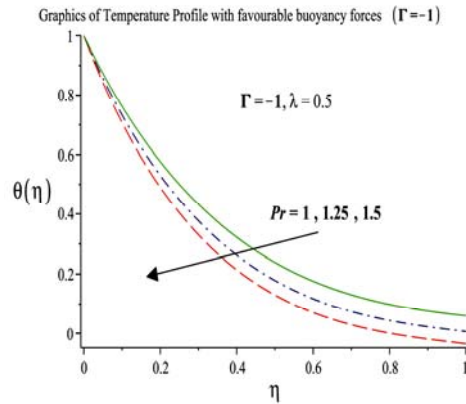


Fig. 5. Prandtl Number effects on temperature profile for eqn. (5) with assisting buoyancy forces.

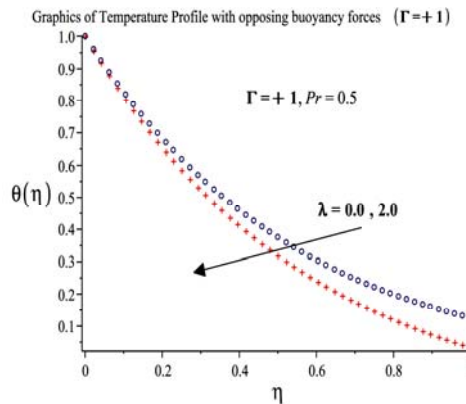


Fig. 6. Marangoni convection parameter effects on temperature profile for eqn. (5) with opposing buoyancy forces.

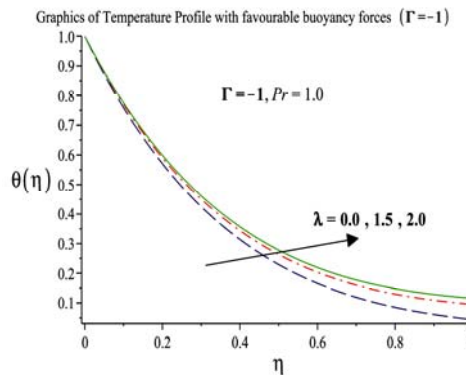


Fig. 7. Marangoni convection parameter effects on temperature profile for eqn. (5) with assisting buoyancy forces.

ACKNOWLEDGEMENTS

The authors would like to thank editor and all referees for their contributions.

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