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Simulation of Fingering Phenomena in Fluid Flow through Fracture Porous Media with Inclination and Gravitational Effect

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ABSTRACT

Here we have studied the fingering phenomena in fluid flow through fracture porous media with inclination and gravitational effect and investigate the applicability of Adomian decomposition method to the nonlinear partial differential equation arising in this phenomena and prove the convergence of Adomian decomposition scheme, which leads to an abstract result and an analytical approximate solution to the equation. Finally developed a simulation result of saturation of wetting phase with and without considering the inclination effect for some interesting choices of parametric data value and studied the recovery rate of the oil reservoir with dimensionless time.

Keywords: Porous media; Adomian decomposition method; Convergence analysis; Simulations.

NOMENCLATURE

A_{mn}	area of a Surface open to flow in the flow direction
F_v	shape function
g	acceleration due to gravity
K	permeability
L_C	characteristic length
l_{ma}	distance from the open surface to the no flow boundary
P_c	capillary pressure
P_o	pressure of oil
P_w	pressure of water
P	mean pressure
R_∞	ultimate recovery
R	recovery
S_w	saturation of water, f raction
S_o	saturation of oil, f raction
t	time
$x = L_c$	dimensionless time
V_{ma}	bulk volume of matrix(core sample
V_w	seepage velocity of water
V_o	seepage velocity of oil
ρ_w	density of water
ρ_o	density of oil
α	inclination of the bed
ϕ	porosity, fraction
μ_w	water viscosity
μ_o	oil viscosity
σ	interfacial tension

1. INTRODUCTION

Fractured hydrocarbon reservoirs are important oil and gas resources. These reservoirs are composed of two continua: the fracture network and matrix. The fractures typically have a high permeability but a very low volume as compared to the matrix whose permeability may be of several orders having lower magnitude but it contains the majority of recover-able oil. Water flooding is frequently implemented to increase recovery in fractured reservoirs. However, the performance of water flooding depends crucially on the wettability of the reservoir. If the reservoir is oil-wet, water will not readily displace oil in the matrix and only the oil in the fractures will be displaced, resulting in poor recoveries and the early water breakthrough. In water-wet fractured reservoirs, imbibition can lead to significant recoveries. Imbibition is the mechanism of displacement of non-wetting phase by wetting phase. Strong capillary forces led to the imbibition of water as the wetting phase into the matrix and the discharged oil is displaced into the fractures. As the viscosity ratio of heavy oil to water is large, viscous forces in the oil phase become dominant and constitute the major factor for controlling flow distortions in the porous formation results perturbation (fingers) which shoot through the porous medium at relatively great speed. Simultaneous occurring of fingering and imbibition leads to fingering phenomena.

Imbibition can take place by cocurrent and/or counter-current flow Scheidegger (1958), Parsons *et al*. (1966), Iffly *et al*. (1972), Hamon *et al*. (1986),Bentsen and Manai (1993), Al-Lawati and Saleh (1996), Pooladi Darvish *et al*. (2000). In cocurrent flow, the water and oil flow in the same direction and water pushes oil out of the matrix. In counter-current flow, the oil and water flow in opposite directions and oil escapes by flowing back in the same direction along which water has imbibed. In Co-current imbibition Fingering imbibition occurs, and it is faster and can be more efficient than counter-current imbibition Verma (1969), Bentsen and Manai (1993), Chimienti *et al*. (1999), Pooladi-Darvish *et al*. (2000) but counter-current imbibition is often the only possible displacement mechanism for cases where a region of the matrix is completely surrounded by water in the fractures Pooladi-Darvish *et al*.(2000), Najurieta *et al*. (2001),Tang *et al*. (2001). Experimentally, this process can be studied by surrounding a core matrix sample with water for measuring the oil recovery as a function of time Iffly *et al*. (1972),Prey *et al*. (1978), Hamon *et al*. (1986),Bentsen and Manai (1993), Cuiec *et al*.(1994), Zhang *et al*. (1995), Cil *et al*. (1998), Chimienti *et al*. (1999), Rangel-German and Kovscek (2002). The imbibition rate is controlled by the permeability of the matrix, its porosity, the oil/water interfacial tension and flow geometry although the ultimate recovery is generally only governed by the residual oil saturation in strongly waterwet systems. Mattax *et al*. (1962), Iffly *et al*. (1972), Hamon *et al*. (1986),Babadagli *et al*. (1992), Al-Lawati and Saleh (1996),Shouxiang *et al*. (1997), Cil *et al*.(1998), Chimienti *et al*. (1999).

Correlations have been developed to predict the recovery from counter-current imbibition as a function of time for different samples. Mattax *et al*. (1962) hypothesized that the oil recovery for systems of different size, shape and fluid properties is a unique function of a dimensionless time. Shouxiang *et al*. (1997) modified an expression derived by Mattax *et al*. (1962) to include the effect of the non-wetting phase viscosity. Their experimental results showed that the imbibition time is inversely proportional to the geometric mean viscosities of water and oil. They proposed the following correlation:

$$T = t\sqrt{\frac{K}{\phi}}\frac{\sigma}{\mu_w \mu_o}\frac{1}{L_c^2} \quad (1)$$

Where T be the dimensionless time, t is time, K is permeability, ϕ is porosity, σ is interfacial tension, μ_w and μ_o are the viscosities of water and oil and L_c is the characteristic length that is determined by the size, shape and boundary conditions of the sample and is defined by Zhang *et al*. (1995) as:

$$F_c = \frac{1}{V_{ma}}\sum_s \frac{A_{ma}}{l_{ma}} \quad (2)$$

$$L_c = \sqrt{\frac{1}{F_c}} \quad (3)$$

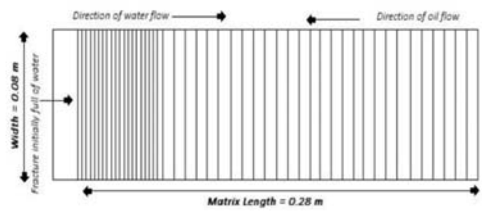

Fig. 1. Grid system for 1-D Simulations of counter-current Imbibition.

Where V_{ma} is the bulk volume of the matrix (core sample), A_{ma} is the area of a surface open to flow in the flow direction, l_{ma} is the distance from the open surface to the no flow boundary and the summation is over all open surfaces of the block.

Mattax *et al*. (1962) showed that recovery as a function of time for a variety of experiments on different water wet samples that fall on a single universal curve as a function of the dimensionless time, T, Eq. (1). In particular, imbibition experimental data presented by Najurieta *et al*. (2001) for Alundum samples and Weiler sandstones, Hamon *et al*. (1986) results for synthetic materials and Zhang *et al*. (1995) results for Berea sandstones with different boundary conditions all scaled onto the same curve that was reasonably well fitted by the following empirical function first proposed by Aronofsky *et al*. (1958):

$$R = R_\infty(1 - e^{-\gamma T}) \quad (4)$$

Where R is the recovery, R_∞ is the ultimate recovery and γ is a constant that best matches the data with a value of approximately 0.5.Eq. (1) was proposed for strongly water-wet media and ignores the effects of wettability and here we have defined the dimensionless time as $T = \left(\frac{gK\rho_w}{\delta_w \phi L_c}\right)$ for studying the recovery rate with dimensionless time.

Furthermore the behaviour of different initial water saturation is a challenge. The presence of initial water saturation reduces capillary pressure, but increases the mobility of invading water. The competition between capillary pressure and relative permeability determines the recovery rate. Bald-win and Spinler (2002) monitored saturation profiles during spontaneous counter current imbibition using magnetic resonance imaging (MRI) they showed a transition from a flat frontal advance to a more gradual water encroachment as the initial water saturation was increased.

A mathematical model that can treat the individual fluid pressures, capillary effects, permeability, porosity and Saturation of wetting phase has been employed in the present work. The Adomain decomposition technique has been adopted for the approximate analytical solution since the problem is computationally intensive. Naturally occurring oil-rich reservoirs to which the present study is applicable are in homogeneous and layered. A qualitative study has been carried out to explore the effect of permeability, porosity, time, distance with saturation of wetting phase and recovery variations with time on flow patterns.

In this paper, we investigate the applicability of Adomian decomposition method to the nonlinear partial differential equation arising in the fingering phenomenon in fluid flow through fracture porous media in order to obtain the analytical approximate solution.

The paper is organized as follows: in section 2, we have written the mathematical model along with some relation and the fundamental equation of the fingering phenomenon is discussed in section 3, In Section 4, the Adomian decomposition method applied to solve nonlinear functional equations. In section 5 where we develop and prove the convergence of Adomian decomposition scheme, which leads to an abstract result and the analytical approximate solution to the equation. Section 6 deals with the simulation result for some interesting choices of initial data. We conclude by summarizing the paper in section 7.

2. MATHEMATICAL MODEL

It is well known that in secondary oil recovery process when water with constant velocity '*V*' is injected into a seam saturated with oil and consisting of homogeneous porous medium, it is assumed that the entire oil on the initial boundary of the seam, $x = 0$ (x is measured in the direction of the displacement), is displaced through a small distance due to the impact of injecting water which forms instability at the common interface where water meets the oil zone. To understand this phenomenon, we consider here a horizontal porous matrix of length L with its impermeable surface filled with oil formatted porous media.

For the definiteness of the problem, consider that there is a uniform water injection into oil saturated porous matrix of an oil formatted region having homogeneous physical characteristics such that the injecting water outs through the oil formation region of the oil reservoir and gives rise to perturberance (fingers) at the interface where injected water pushes the oil from the oil formatted region. This furnishes well-developed fingers as in Figs. 2 and 3. The stability of a water flood depends on the mobility ratio between oil and water, heterogeneity of the porous medium, segregation of fluids in the reservoir, and dissipation of fluid fronts caused by capillary pressure. Instabilities occurs in both miscible and immiscible processes and originate on the interface between oil and water. These frontal instabilities are often characterized by a number of penetrating fingers of displacing fluid. Therefore, the entire oil at the initial boundary $x = 0$ (x being measured in the direction of displacement) is displaced through a distance " L_c " due to water injection. It is further assumed that complete saturation exists at the initial boundary, and the saturation of displaced water (fingers) in the oil zone may happen up to distance $x = L_c$.

Relative Permeability and Phase Saturation Relation:

For definitions of mathematical analysis, we assume a standard form for the relationship between capillary pressure, permeability of water and permeability of oil with phase saturation as:

$$K_w = S_w, K_o = 1 - S_w, P_c = -\beta S_w \quad (5)$$

Where β is constant.

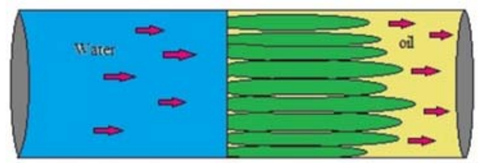

Fig. 2. Representation of Fingering in a cylindrical piece of homogeneous porous media.

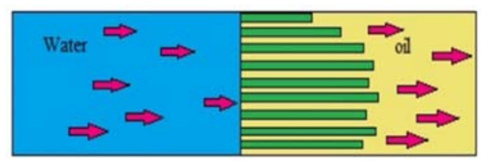

Fig. 3. Schematic presentation of the fingering (instability) phenomenon.

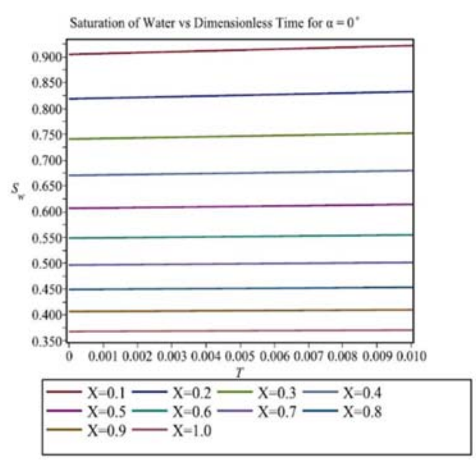

Fig. 4. Saturation of Water vs Time for $\alpha = 0^\circ$.

3. FUNDAMENTAL EQUATION

From Darcy's law, the seepage velocity of water V_w

and oil V_o can be written as

$$V_w = \frac{-K_w}{\delta_w} K\left(\frac{\partial P_w}{\partial x} + \rho_w g \sin\alpha\right) \tag{6}$$

$$V_o = \frac{-K_o}{\delta_o} K\left(\frac{\partial P_o}{\partial x} + \rho_o g \sin\alpha\right) \tag{7}$$

Where “K” is permeability of homogenous medium, K_o and K_w are relative permeability of oil and water which are function of So and $S_w\rho_w$ and ρ_w are constant densities of water and oil respectively, while δ_w and δ_w are constant kinematic viscosity of the phases in homogenous porous media, α is the inclination of the bed, g is acceleration due to gravity.

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{8}$$

$$\phi\frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{9}$$

Where “ ϕ ” is the porosity of medium. from the definition of phase saturation, we have

$$S_w + S_o = 0 \tag{10}$$

The capillary pressure, which is defined as the pressure discontinuity of the following phases across the common interface is written as

$$P_c = P_o - P_w \tag{11}$$

Impies

$$\frac{\partial P_w}{\partial x} = \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \tag{12}$$

Equation of motion for saturation can be obtained by substituting the values of V_w and V_o from Eqs. (6) and (7) in Eqs. (8) and (9) respectively. Thus, we have

$$\phi\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{K_w}{\delta_w}K\right)\left(\frac{\partial P_w}{\partial x} + \rho_w g \sin\alpha\right)\right] \tag{13}$$

$$\phi\frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{K_o}{\delta_o}K\right)\left(\frac{\partial P_o}{\partial x} + \rho_o g \sin\alpha\right)\right] \tag{14}$$

Equation (12) and Equation (13) together

$$\phi\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{K_w}{\delta_w}K\right)\left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} + \rho_w g \sin\alpha\right)\right] \tag{15}$$

Now by considering Eqs. (10),(14) and (15), it becomes

$$\left[\frac{\partial P_o}{\partial x}K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right) - \frac{K_w}{\delta_w}k\frac{\partial P_c}{\partial x} + g\sin\alpha K\left(\frac{K_o}{\delta_o}\rho_o + \frac{K_w}{\delta_w}\rho_w\right)\right] = -q \tag{16}$$

where q is constant of integration. Eq. (16) Implies

$$\frac{\partial P_o}{\partial x} = \left[\frac{-q + \frac{K_w}{\delta_w}k\frac{\partial P_c}{\partial x}}{K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)} - \frac{g\sin\alpha K\left(\frac{K_o}{\delta_o}\rho_o + \frac{K_w}{\delta_w}\rho_w\right)}{K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)}\right] \tag{17}$$

Equations (15) and (17) together gives

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\frac{K.K_o}{\delta_o}\left(\frac{\partial P_c}{\partial x} + g\sin\alpha\left(\rho_o - \rho_w\right)\right)}{\left(1 + \frac{K_o.\delta_w}{K_w.\delta_o}\right)} + \frac{q}{\left(1 + \frac{K_o.\delta_w}{K_w.\delta_o}\right)}\right] = 0 \tag{18}$$

The value of the pressure of oil (P_o) can be written as

$$P_o = \overline{P} + \frac{1}{2}P_c \Rightarrow \frac{\partial P_o}{\partial x} = \frac{1}{2}\frac{\partial P_c}{\partial x} \tag{19}$$

where $\overline{P}$ is the mean pressure.

Equations (16),(19) together with Eq. (18) gives,

$$\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\left(\frac{K_w}{\delta_w}K\right)\left(\frac{1}{2}\frac{\partial P_o}{\partial x} - g\rho_w\sin\alpha\right)\right] = 0 \tag{20}$$

on using equation (5) in equation (20), it reduces to

$$\frac{\partial S_w}{\partial t} - \frac{\partial}{\partial x}\left[\left(\frac{S_w}{\delta_w}K\right)\left(\frac{\beta}{2}\frac{\partial S_w}{\partial x} + g\rho_w\sin\alpha\right)\right] = 0 \tag{21}$$

Equation (21) describe the equation of motion for saturation of wetting phase in fingering imbibition in fluid flow through fracture porous media with inclination and gravitational effect.

Using dimensionless variables, Eq. (21) reduces to,

$$X = \frac{x}{L_c} \quad T = \left(\frac{gK\rho_w}{\delta_w\phi L_c}\right)t$$

$$\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial X}\left(S_w\frac{\partial S_w}{\partial X}\right) + \sin\alpha\frac{\partial S_w}{\partial X} \tag{22}$$

Here we choose appropriate initial and Dirichlets boundary condition due to the behaviour of saturation of displaced water at the interface in instability phenomena; that is, instability of oil and water zone at the interface is high, and it becomes stable as it becomes away from the interface and by Verma (1969) as,

$$S_w(X,0)=e^{-X},\quad S_w(0,T)=f_1(T),\quad S_w(1,T)=f_2(T) \tag{23}$$

where f_1 and f_2 are the saturation of water at common interface X = 0 and saturation of water at the end of the matrix of length X = 1 (i.e. $X = L_c$). Here, during fingering phenomena, saturation fingers may take place up to the end of matrix, that is, up to $X = L_c$. To stabilize or to find the behaviour of the saturation fingers, it is necessary to discuss the behaviour of saturation of displace water by solving (22) together with (23). Eq. (22) is the desired nonlinear partial differential equation with suitable initial and boundary conditions which describes the saturation of displaced water in fingering phenomena arising during the oil recovery process.

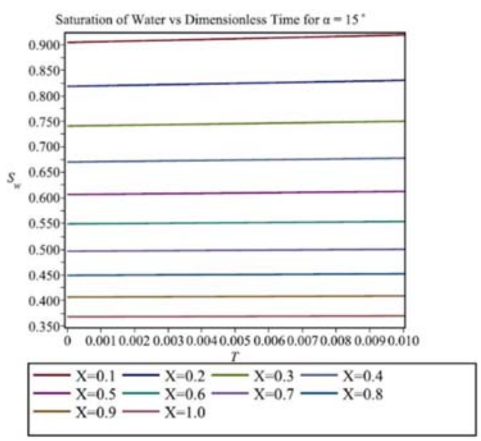

Fig. 5. Saturation of Water vs Time for $\alpha = 15^\circ$.

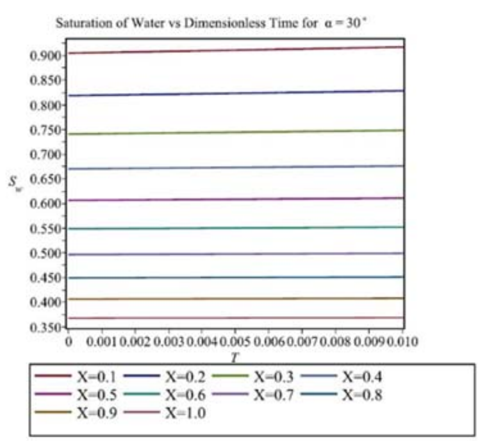

Fig. 6. Saturation of Water vs Time for $\alpha = 30^\circ$.

4. Analysis of the Adomian Decomposition Method

In the early 1980s, a new numerical method was developed by Adomian (1994) in order to solve non-linear functional equations of the form

$$LS_w + RS_w + NS_w = g \tag{24}$$

Using an iterative decomposition scheme that led to elegant computation of closed-form analytical solutions or analytical approximations to solutions. In (24), L represents the linear part, N represents the nonlinear part, R represents the remainder or lower order terms and g is non homogeneous right-hand side. The solution S_w and non linearity N are assumed to have the following analytic expansions respectively,

$$S_w = \sum_{n=0}^{\infty} S_{wn}, \qquad NS_w = \sum_{n=0}^{\infty} A_n \tag{25}$$

Where $A's$ the are the Adomian polynomials that depend only on $S_{w0}, S_{w1}, S_{w2}, \ldots, S_{wn}$ and are given by the following formula:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^k S_{wk}\right)\right], \quad n \ge 0 \tag{26}$$

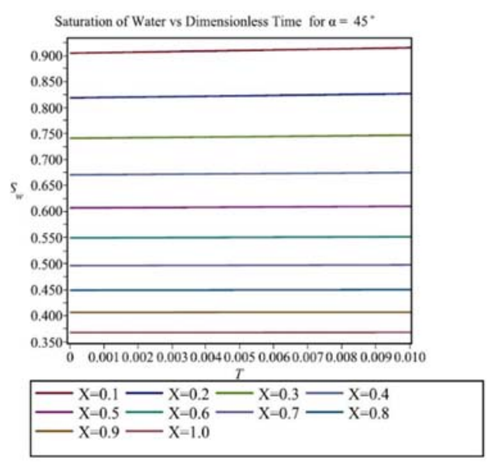

Fig. 7. Saturation of Water vs Time for $\alpha = 45^\circ$.

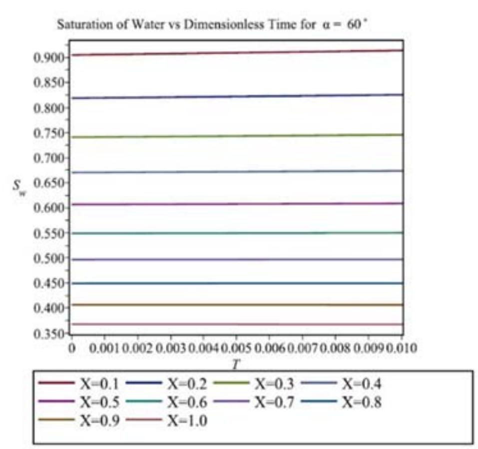

Fig. 8. Saturation of Water vs Time for $\alpha = 60^\circ$.

Table 1 Parametric value of the Parameters

Parametric	Unit	Network Modelling Data
Porosity	Frac	0.5
Permeability	m^2	10^{-4}
Water density	Kgm^{-3}	1010
Water viscosity	Pas	$0.967*10^{-3}$
Acceleration due to gravity	m/s^2	0.101972

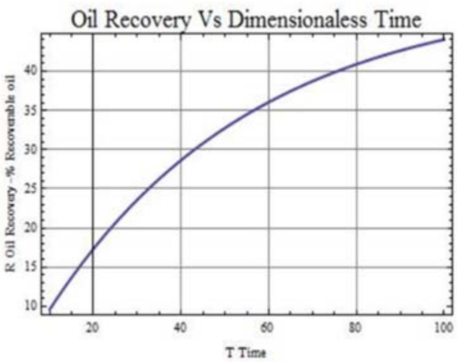

Fig. 9. Normalized Oil Recovery rate with Dimensionless Time

In order to better explain the method, we will first assume the convergence of the series in (25) and deal with the rigorous convergence issues later. The parameter *k* is a dummy variable introduced for ease of computation. There are several different versions of (26) that can be found in the literature that leads to easier computation of the $A_n's$. It should be noted that the $A_n's$ are the terms of analytic expansion of NS_w , where $S_w = \sum_{n=0}^{\infty} S_{wn}$.

In Adomian (1994), has shown that the expansion for NS_w in eq. (25) is a rearrangement of the Taylor series expansion of NS_w about the initial function S_{w0} in a suitable Hilbert or Banach space. Substitution of (25) in (24) results in the following,

$$L\left(\sum_{n=0}^{\infty} S_{wn}\right) = -R\left(\sum_{n=0}^{\infty} S_{wn}\right) - \sum_{n=0}^{\infty} A_n + g \tag{27}$$

The above equation can be rewritten in a recursive fashion, yielding iterates S_{wn} , the sum of which converges to the solution S_w satisfying (27) if it exists,

$$\sum_{n=0}^{\infty} S_{wn} = L^{-1}R\left(\sum_{n=0}^{\infty} S_{wn}\right) - L^{-1}\sum_{n=0}^{\infty} A_n + L^{-1}g$$
$$S_{w0} = L^{-1}g; S_{w,n+1} = L^{-1}R(S_{wn}) - L^{-1}(A_n) \tag{28}$$

Typically, the symbol L^{-1} represents a formal inverse of the linear operator *L*.

The objective of the decomposition method is to make possible physically realistic solutions of complex systems without the usual modeling and solution compromises to achieve tractability. It essentially combines the fields of ordinary and partial differential equations. The ADM decomposes a solution in to an infinite series which converges rapidly to the exact solution. The convergence of the ADM has been investigated by a number of authors (Abbaoui and Cherruault (1994),Cherruault (1989),Cherruault and Adomian (1993). This method can be applied directly for all types of differential and integral equations, linear or nonlinear, with constant or variable coefficients. The nonlinear problems are solved easily and elegantly without linearizing the problem by using ADM. The technique is capable of greatly reducing the size of computational work while still it provides an efficient numerical solution with high accuracy. It also avoids linearization, perturbation and discretization unlike other classical techniques.

5. CONVERGENCE ANALYSIS OF THE ADOMIAN DECOMPOSITION METHOD

We recall the following theorem from Mavoungou and Cherruault (1992) which guarantees the convergence of Adomians method for the general operator equation given by $LS_w + RS_w + NS_w = g$.

Consider the Hilbert space $H = L^2((\alpha,\beta)\times[0,T])$ defined by the set of applications $S_w: \ (a,\beta)\times[0,T] \to R$ with

$$\int_{(a,\beta)\times[0,T]} S_w^2(\eta,\xi) < +\infty \tag{29}$$

Let us denote,

$$LS_w = \frac{\partial S_w}{\partial T}, NS_w = \frac{\partial}{\partial X}\left(S_w \frac{\partial S_w}{\partial X}\right), RS_w = \sin\alpha\left(\frac{\partial S_w}{\partial X}\right)$$

$$TS_w = NS_w + RS_w = \frac{\partial}{\partial X}\left(S_w \frac{\partial S_w}{\partial X}\right) + \sin\alpha\left(\frac{\partial S_w}{\partial X}\right) \tag{30}$$

Theorem 1. Let $TS_w = -RS_w - NS_w$ be a hemi – continuous operator in Hilbert Space *H* and satisfy the following,

Hypothesis(H_1) :

$$(TS_{w1} - TS_{w2}, S_{w1} - S_{w2}) \geq k\|S_{w1} - S_{w2}\|^2 \quad k > 0, \forall S_{w1}, S_{w2} \in H \tag{31}$$

Hypothesis (H_2) :Whatever may be $M > 0$, there exit constant $C(M) > 0$ such that for $S_{w1}, S_{w2} \in H$ with

$\|S_{w1}\| \leq M, \|S_{w2}\| \leq M$, we have

$$(TS_{w1} - TS_{w2}, w) \leq C(M)\|S_{w1} - S_{w2}\| \, \|w\| \quad forevery \;\; w < H \tag{32}$$

Then, for every $g \in H$,the nonlinear functional equation $LS_w + RS_w + NS_w = g$ admit a unique solution $S_w \in H$ Furthermore, if the solution S_w can be represented in a series form given by $S_w = \sum_{n=0}^{\infty} S_{wn}\lambda^n$, then the Adomian decomposition scheme corresponding to the functional equation under consideration converges strongly to $S_w \in H$, which is the unique solution to the functional equation.

proof: verification of hypothesis(H_1)

$$\left(TS_{w1} - TS_w\right) = -\frac{1}{2}\frac{\partial^2}{\partial X^2}(S_{w1}^2 - S_{w2}^2) - \sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2})$$

$$(TS_{w1} - TS_{w2}, S_{w1} - S_{w2}) = \left(-\frac{1}{2}\frac{\partial^2}{\partial X^2}(S_{w1}^2 - S_{w2}^2), S_{w1} - S_{w2}\right) + \left(-\sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2}), S_{w1} - S_{w2}\right) \tag{33}$$

Since $\frac{\partial^2}{\partial X^2}$ is a differential operator in H, then there exit a constant "δ" such that,

According to Schwartz inequality, we get

$$\left(\frac{1}{2}\frac{\partial^2}{\partial X^2}(S_{w1}^2 - S_{w2}^2)\right) \leq \frac{1}{2}\delta\left\|S_{w1}^2 - S_{w2}^2\right\| \, \left\|S_{w1} - S_{w2}\right\| \tag{34}$$

$$\left(\sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2}), S_{w1} - S_{w2}\right) \leq \sin\alpha\sigma\|S_{w1} - S_{w2}\| \, \|S_{w1} - S_{w2}\| \tag{35}$$

Now, by using mean value theorem, then we have

$$\left(-\sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2}), S_{w1} - S_{w2}\right) \geq \sin\alpha\sigma\|S_{w1} - S_{w2}\|^2 \tag{36}$$

For $\|S_{w1}\| \leq M$ and $\|S_{w2}\| \leq M$

Therefore

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial X^2}(S_{w1}^2 - S_{w2}^2), S_{w1} - S_{w2}\right) \geq \delta M\|S_{w1} - S_{w2}\|^2 \tag{37}$$

$$\left(-\sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2}), S_{w1} - S_{w2}\right) \geq \sin\alpha\sigma\|S_{w1} - S_{w2}\|^2 \tag{38}$$

Substituting (37) and (38) in (33),

$$(TS_{w1} - TS_{w2}, S_{w1} - S_{w2}) \geq k\|S_{w1} - S_{w2}\|^2 \tag{39}$$

Where $k = \delta M + \sin\alpha\sigma M$ **.**Hence we find the Hypothesis (H_1). For Hypothesis (H_2),

$$\begin{aligned}&\left(TS_{w1} - TS_{w2}, w\right) = \\ &\left(\left(-\frac{1}{2}\frac{\partial^2}{\partial X^2}(S_{w1}^2 - S_{w2}^2) - -\sin\alpha\frac{\partial}{\partial X}(S_{w1} - S_{w2})\right), w\right) \\ &\leq \delta M \sin\alpha\sigma\|S_{w1} - S_{w2}\|^2\|w\| \\ &= C(M)\|S_{w1} - S_{w2}\|\|w\|\end{aligned} \tag{40}$$

Where $C(M) = \delta M + \sin\alpha\sigma$ and therefore hypothesis (H_2) holds.

6. SIMULATIONS RESULT

ADM in T - direction

Using the analysis of Adomian Decomposition Method, (21) can be written in operator form $L_T S_w$ as

$$L_T S_w(X,T) = L_X(NS_w(X,T)) + \sin\alpha L_X S_w \tag{41}$$

Operating inverse operator on both sides of Eq. (41), it gives

$$S_w(X,T) = S_{w0}(X) + L_T^{-1}(L_X(NS_w(X,T)) + \sin\alpha L_X S_w) \tag{42}$$

where $NS_w(X,T) = S_w\left(\frac{\partial S_w}{\partial X}\right)$ and $S_{w0}(X)$ can be solved subject to the corresponding initial condition $S(X,0) = f(X) = e^{-X}$. It is well known form (25) that the solution of (21) can be written in the series form as follows

$$S_w(X,T) = \sum_{n=0}^{\infty} S_{wn}(X,T) \tag{43}$$

where $S_{w1}, S_{w2}, S_{w3}...$ are the saturation of the different fingers at any distance X and any time $t > 0$ and the non linear term can be represented as $NS_w(X,T) = \sum_{n=0}^{\infty} A_n$, where $A_n's$ are Adomian's special polynomials to be determined and defined by (26).

Following the analysis of Adomian decomposition

Table 2 Saturation vs. Time for Distance Fixed at X=0.1

X=0.1					
T/X	$\alpha=0^0$	$\alpha=15^0$	$\alpha=30^0$	$\alpha=45^0$	$\alpha=60^0$
T=0.001	0.9065	0.9062	0.9060	0.9058	0.9057
T=0.002	0.9081	0.9077	0.9072	0.9069	0.9066
T=0.003	0.9098	0.9091	0.9084	0.9079	0.9074
T=0.004	0.9115	0.9105	0.9097	0.9089	0.9083
T=0.005	0.9132	0.9120	0.9109	0.9099	0.9092
T=0.006	0.9149	0.9135	0.9121	0.9110	0.9101
T=0.007	0.9166	0.9150	0.9134	0.9121	0.9110
T=0.008	0.9184	0.9165	0.9147	0.9131	0.9119
T=0.009	0.9201	0.9180	0.9159	0.9142	0.9129
T=0.010	0.9219	0.9194	0.9172	0.9153	0.9138

Table 3 Saturation vs. Time for Distance Fixed at X=0.2

X=0.2					
T/X	$\alpha=0^0$	$\alpha=15^0$	$\alpha=30^0$	$\alpha=45^0$	$\alpha=60^0$
T=0.001	0.8201	0.8199	0.8197	0.8195	0.8194
T=0.002	0.8214	0.8210	0.8206	0.8203	0.8200
T=0.003	0.8228	0.8222	0.8216	0.8210	0.8207
T=0.004	0.8242	0.8233	0.8225	0.8225	0.8213
T=0.005	0.8256	0.8245	0.8235	0.8226	0.8220
T=0.006	0.8270	0.8257	0.8245	0.8234	0.8226
T=0.007	0.8284	0.8268	0.8254	0.8242	0.8233
T=0.008	0.8298	0.8280	0.8264	0.8250	0.8240
T=0.009	0.8312	0.8293	0.8274	0.8259	0.8247
T=0.010	0.8327	0.8305	0.8284	0.8267	0.8254

Table 4 Saturation vs. Time for Distance Fixed at X=0.3

X=0.3					
T/X	$\alpha=0^0$	$\alpha=15^0$	$\alpha=30^0$	$\alpha=45^0$	$\alpha=60^0$
T=0.001	0.7419	0.7417	0.7415	0.7414	0.7413
T=0.002	0.7430	0.7426	0.7423	0.7420	0.7417
T=0.003	0.7441	0.7436	0.7430	0.7426	0.7422
T=0.004	0.7453	0.7445	0.7438	0.7432	0.7427
T=0.005	0.7464	0.7454	0.7445	0.7437	0.7432
T=0.006	0.7475	0.7464	0.7453	0.7443	0.7436
T=0.007	0.7487	0.7473	0.7460	0.7450	0.7441
T=0.008	0.7498	0.7483	0.7468	0.7456	0.7446
T=0.009	0.7510	0.7492	0.7476	0.7462	0.7451
T=0.010	0.7522	0.7502	0.7484	0.7468	0.7456

method as discussed in Abbaoui and Cherruault (1994),Gabet (1994) for the determination of the components $S_{wn}(X,T)$ of $S_w(X,T)$,we set the recursive relation as

$$\sum_{n=0}^{\infty} S_{wn}(X,T) = e^{-X}$$
$$L_T^{-1}\left[L_X\left(\sum_{n=0}^{\infty} A_n\right) + \sin\alpha L_X S_w\right] \tag{44}$$

Table 5 Saturation vs. Time for Distance Fixed at X=0.4

X=0.4					
T/X	α=0⁰	α=15⁰	α=30⁰	α=45⁰	α=60⁰
T=0.001	0.6712	0.6710	0.6709	0.6707	0.6706
T=0.002	0.6721	0.6718	0.6715	0.6712	0.6710
T=0.003	0.6730	0.6725	0.6720	0.6713	0.6713
T=0.004	0.6740	0.6733	0.6726	0.6720	0.6716
T=0.005	0.6749	0.6740	0.6732	0.6725	0.6719
T=0.006	0.6758	0.6748	0.6738	0.6729	0.6723
T=0.007	0.6767	0.6755	0.6744	0.6734	0.6726
T=0.008	0.6777	0.6763	0.6750	0.6738	0.6730
T=0.009	0.6786	0.6770	0.6756	0.6743	0.6733
T=0.010	0.6796	0.6778	0.6762	0.6747	0.6736

Table 6 Saturation vs. Time for Distance Fixed at X=0.5

X=0.5					
T/X	α=0⁰	α=15⁰	α=30⁰	α=45⁰	α=60⁰
T=0.001	0.6073	0.6071	0.6070	0.6068	0.6067
T=0.002	0.6080	0.6077	0.6074	0.6071	0.6070
T=0.003	0.6088	0.6083	0.6078	0.6075	0.6072
T=0.004	0.6095	0.6089	0.6083	0.6078	0.6074
T=0.005	0.6103	0.6095	0.6087	0.6081	0.6076
T=0.006	0.6110	0.6101	0.6092	0.6084	0.6078
T=0.007	0.6118	0.6107	0.6096	0.6087	0.6081
T=0.008	0.6125	0.6113	0.6101	0.6091	0.6083
T=0.009	0.6133	0.6119	0.6105	0.6094	0.6085
T=0.010	0.6141	0.6125	0.6110	0.6097	0.6087

Where $S_{w0} = e^{-X} = S_w(X,0)$ from the initial condition and $S_{w,k+1} = L_T^{-1}((A_k)_X + \sin\alpha\left(S_w\right)_X)$. In view of the Eq. (44), the approximate solution in the series form is given by

$$S_w(X,T) = S_{w0} + S_{w1} + S_{w2} + S_{w3} + S_{w4} + S_{w5} + S_{w6} + S_{w7} + S_{w8} + S_{w9} + \ldots$$

$$= \Bigg(-\frac{1530966437}{540}\sin^3\alpha e^{-7X} - \frac{1941781969}{280}\sin\alpha e^{-9X} + \frac{1238328}{5}\sin^4\alpha e^{-6X} - \frac{5971984384}{540}\sin^2\alpha e^{-8X} - \frac{1}{362880}\sin^9\alpha e^{-X} - \frac{1796875}{144}\sin^5\alpha e^{-5X} + \frac{4}{315}\sin^8\alpha e^{-2X} - \frac{2187}{560}\sin^7\alpha e^{-3X} + \frac{45056}{135}\sin^6\alpha e^{-4X} + \frac{61239884500}{567}e^{-10X}\Bigg)T^9 + \Bigg(\frac{1}{40320}\sin^8\alpha e^{-X} - \frac{16}{315}\sin^7\alpha e^{-2X} + \frac{729}{80}\sin^6\alpha e^{-3X} - \frac{22528}{45}\sin^5\alpha e^{-4X} - \frac{1492996096}{315}\sin\alpha e^{-8X} + \frac{218709491}{180}\sin^2\alpha e^{-7X} + \frac{1796875}{144}\sin^4\alpha e^{-5X} - \frac{825552}{5}\sin^3\alpha e^{-6X} + \frac{215753552}{280}e^{-9X}\Bigg)T^8 + \Bigg(-\frac{1}{5040}\sin^7\alpha e^{-X} + \frac{8}{45}\sin^6\alpha e^{-2X} - \frac{31244213}{90}\sin\alpha e^{-7X} + \frac{412776}{5}\sin^2\alpha e^{-6X} - \frac{359375}{36}\sin^3\alpha e^{-5X} + \frac{5632}{9}\sin^4\alpha e^{-4X} - \frac{729}{40}\sin^5\alpha e^{-3X} + \frac{186624512}{315}e^{-8X}\Bigg)T^7 + \Bigg(\frac{1}{720}\sin^6\alpha e^{-X} + \frac{71875}{12}\sin^2\alpha e^{-5X} - \frac{5632}{9}\sin^3\alpha e^{-4X} - \frac{137592}{5}\sin\alpha e^{-6X} + \frac{243}{8}\sin^4\alpha e^{-3X} - \frac{8}{15}\sin^5\alpha e^{-2X} + \frac{4463459}{90}e^{-7X}\Bigg)T^6$$

$$+\left(-\frac{1}{120}\sin^5\alpha e^{-X}+\frac{4}{3}\sin^4\alpha e^{-2X}+\frac{1408}{3}\sin^2\alpha e^{-4X}\right.$$
$$-\frac{14375}{6}\sin\alpha e^{-5X}-\frac{81}{2}\sin^3\alpha e^{-2X}-\frac{704}{3}\sin\alpha e^{-4X}$$
$$\left.+\frac{81}{2}\sin^2\alpha e^{-3X}+\frac{2875}{6}e^{-5X}\right)T^4+\left(-\frac{1}{6}\sin^3\alpha e^{-X}+\right.$$
$$\left.4\sin^2\alpha e^{-2X}-27\sin\alpha e^{-3X}+\frac{176}{3}e^{-4X}\right)T^3$$
$$\left(\frac{1}{2}\sin^2\alpha e^{-X}+4\sin\alpha e^{-2X}+9e^{-3X}\right)T^2 \qquad (45)$$
$$+(-\sin\alpha e^{-X}+2e^{-2X})T+e^{-X}+....$$

ADM in X direction

We can use the boundary condition if we proceed in X - direction which gives constant solution provided f_1 and f_2 are constant and gives trivial solution if f_1 and f_2 are zero.

From Figs. 4,5,6,7 and 8 it is apparent that Saturation of water increases as time increases at a fixed point X = 0.1, X = 0.2, X = 0.3, X = 0.4 and X = 0.5 for inclination , $\alpha=0^\circ$ α = 15◦, $\alpha=30^\circ$, $\alpha=45^\circ$, $\alpha=60^\circ$ and subsequently decreases with time as inclination of the bed increases results decrease in saturation and recovery rate. Hence it may conclude that the saturation of wetting phase increases with time for zero inclination and small inclination results increase the recovery rate of the oil reservoir but as the inclination increases it results lower the saturation rate implies less recovery rate of the oil reservoir.

7. CONCLUSION

Convergence of the Adomian decomposition scheme for the case of fingering phenomena has been proved and studied the variation of saturation of water in *X* and *T* direction for particular parametric values of the parameter in dimensionless form. Eq. (45) represents the saturation of wetting phase for fingering phenomena with the inclination, dimensionless time and distance. Table (2,3,4,5,6) and graph (4,5,6,7,8) shows that the saturation of wetting phase be maximum for zero inclination implies more recovery rate with time. Similarly as the inclination of the field increases, saturation rate be minimum results less recovery rate, which is physically consistent with the real word phenomena.

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