Journal of Applied Fluid Mechanics, Vol. 13, No. 6, pp. 1807-1817, 2020.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.47176/jafm.13.06.31157

Study of Non-Newtonian Blood Flow of Jeffrey Fluid in an Elastic Tube

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(Received November 13, 2019; accepted May 7, 2020)

ABSTRACT

Most of the biological ducts which considered to be elastic in nature are layered and possess different fluid properties from that of pumped fluids Best & Taylor (1958). A mathematical model is presented according to the two-layered blood flow in an artery. Such flow demands a two-fluid model with elastic boundary. In addition, biofluids such as blood can be described well using two-fluid models rather than single fluid model. In the present paper, the flow of a Jeffrey fluid in contact with a Newtonian fluid is considered. The expressions for velocity of core and peripheral fluids and the flux flow rate are derived. The effect of the peripheral layer on the fluid motion and pumping characteristics is presented. The core and peripheral fluid velocities along with interface velocity are obtained in terms of inlet, outlet and external pressure; Jeffrey parameter, ratio of viscosity and elastic properties. The results obtained from the present analytical study of flux variation considering elastic properties are in good comparison with the published literature. It is concluded that the elastic parameters $(t_1 \& t_2)$ significantly affect velocity and flux. Further, it is also found that flux reduces with increase in viscosity ratio. The analysis of the interface velocity on various physical parameters may be useful in understanding the behavior of the blood flow in normal and pathological states.

Keywords: Jeffrey fluid; Non-Newtonian fluid; Elastic tube; Blood flow.

NOMENCLATURE

L	length of the tube
p_2	outlet pressure
P	pressure gradient
p	pressure of the fluid
p_0	external pressure
p_1	inlet pressure
r	radial coordinate
$T(a)$	tension of the tube
t_1, t_2	elastic parameters
U	average velocity
u_1, u_2	the fluid velocities in core and peripheral regions
u_0	interface velocity
z	axial coordinate
μ_1, μ_2	core and peripheral layer viscosities
λ_1	Jeffrey parameter
θ	azimuthal angle
σ	conductivity of the tube
a_0	radius of the inelastic tube
$a(z)$	radius of the tube
$\tau = \frac{\tau_2}{\tau_1}$	ratio of shear stresses
ψ_1, ψ_2	stream functions in core and peripheral regions
a_1, a_2	core and peripheral radius of the tube
μ	the ratio of viscosity of peripheral layer to the core layer
Q	total volume flux
Q_1, Q_2	volumetric flow rate in the core and peripheral regions

1. INTRODUCTION

Most of the earlier works on biofluid flow in physiological systems were modelled as viscous flow in rigid tubes. The present study deals with the biofluid flow in a flexible tube incorporating the elastic effects of the bounding wall. Newtonian and non-Newtonian fluids are considered for the study in biofluid transport problems for various conditions of the biological system. The constitution of biofluids such as blood demands a two-fluid model rather than a single fluid model. The blood vessel being elastic further signifies the rheological behavior of blood flow. These fluids are modelled as Newtonian/ non-Newtonian fluids. Some of biofluids like water, air, chyme etc. can modelled as a single component fluid. When the stress in the biofluid is not proportional to rate of strain (may be pathological state), it requires modelling through non-Newtonian fluid. There exists resistance to flow of blood in large and small arteries and is dependent on Reynolds number. In capillary blood vessels, Re is of order 10^{-2} or smaller. Hence the flow is controlled mainly by the viscous force and pressure. Apparent coefficient of viscosity of fluid is take into account due to the interaction of blood cells, plasma with the vessel wall. This coefficient of viscosity increases because leucocytes adhere to the wall of the blood vessel and thereby increasing resistance to blood flow. Also, the thrombocytes may be activated, causing clotting and hence increasing resistance. In such cases, single fluid model cannot describe the physical nature of the blood flow. In addition, some of the biological ducts are coated with a different fluid during biofluid pumping. These physiological aspects demand a two-fluid model for the study of biofluid flow (for example blood) in biological systems.

Guyton (1971) noticed the existence of a mucus layer in the intestinal flow and similar peripheral layers in biofluid systems. Bugliarello and Sevilla (1970) and Cokelet (1972) concluded that blood flow in small vessels is a two layered model due to the experimental finding signifies core layer which is non-Newtonian containing suspension of erythrocytes and the peripheral layer with plasma is a Newtonian fluid. Chaturani and_Kaloni (1976) investigated a two layered Poiseuille model for blood flow in arteries with couple stress fluid as a special case. Srivastava and Srivastava (1982) studied the peristaltic transport of a two fluid model in a non-uniform channels and tubes. Brasseur *et al.* (1987) investigated the peristaltic transport in two immiscible viscous fluids in a channel. Srivastava and Saxena (1994) investigated two-layered Casson fluid flow through the artery with mild stenosis. It has been found that the wall shear stress and flow resistance decreases as the peripheral layer viscosity decreases. Also, the unsteady flow of two immiscible conducting fluids between two permeable beds is examined by Vajravelu *et al.* (1995). The pulsatile unsteady two layered blood flow in a stenosed flexible artery due to peripheral layer viscosity is numerically studied by Chakravarty *et al.* (2004). Srivastava (2007) observed the simultaneous effects of hematocrit and the peripheral layer on the flow characteristics in small blood vessels. Santhosh and Radhakrishnamacharya (2016) analyzed a two fluid model in which the peripheral region consists of Newtonian fluid and Core region with Herschel-Bulkley fluid and noticed that the flow exhibits the anomalous Fahraeus - Lindvist effect. Buradi and Mahalingam (2018) were made investigations on two layered model of blood flow through stenosed arteries. Several researchers (see Haldar and Anderson (1996), Ponalagusamy (2007), Rekha and Usha (2011), Sankar (2012), Hazarika and Sharma (2014), presented the two phase model with central core containing suspended erythrocyte and cell free layer surrounding the core in the presence of magnetic field. It is reported that an appropriate values of magnetic field regulates axial velocities and effective viscosity. Ramachandra Rao and Usha (1995) investigated the studies on a circular tube. They reported that the reflux occurs in the entire pumping range for all viscosity ratios and it is absent in the entire range of copumping.

The earlier studies deal with the biofluid flow in a flexible tube incorporating the elastic effects of the bounding wall. Roach and Burton (1957) experimentally measured the static pressure - volume curve as tension versus elongation of human external iliac artery. It is clearly mentioned that the distensibility of shape of arteries is attributed due to its elastic nature. Rubinow and Keller (1972) analyzed the viscous fluid flow through elastic tubes by considering blood flow applications. Several models were proposed by Kapur (1985) to explain the above said phenomenon. Further, Vajravelu *et al.* (2011b) presented model for Herschel- Bulkley fluid through an elastic tube with uniform cross section. Sreenadh and Devaki (2012) focused the behavior of peristaltic pumping of viscous fluid in an elastic tube. Vajravelu *et al.* (2016) studied the peristaltic transport of a Casson fluid in an elastic tube. The effect of variation in tube radius and wave amplitudes were studied and found to be better using Rubinow and Keller method as compared that of Mazumdar method for yield stress and the amplitude ratio. Srinivas *et al.* (2017) adopted perturbation technique and studied the effects of Carreau fluid in an elastic tube. A good comparison for volumetric flow rate using Rubinow and Keller, Mazumdar model were found.

The Jeffrey model is the simplest model involving time derivatives which exhibits the characteristics of non-Newtonian fluid and is accepted as a model for blood by many investigators (See Akbar *et al.* (2013), Kothandapani and Srinivas (2008), Vajravelu *et al.* (2016), Sreenadh *et al.* (2017), Hayat *et al.* (2017), Elbanhawy *et al.* (2019)).

According to the available literature most of the researchers dealt with two layered fluid flows in inelastic tubes. The two-layered model enables various combinations of choosing the core and peripheral regions through the viscosity ratios of the fluids. In view of these facts the authors study the flow of two immiscible fluids through an elastic tube. The core region is filled with Jeffrey fluid whereas the peripheral region is occupied by Newtonian fluid. Analytical expressions for flux, velocity, interface and stream functions are derived and discussed in detail through graphs. The interface velocity is obtained to understand the behavior of blood flow in an artery.

2. Mathematical Formulation

The constitutive equations for an incompressible Jeffrey fluid are (Vajravelu *et al*. (2016a))

$$\tau = -pI + S \tag{1}$$

$$S = \frac{\mu_1}{1+\lambda_1}\left(\dot{\gamma} + \lambda_2 \ddot{\gamma}\right) \tag{2}$$

where τ and S represent Cauchy stress tensor and extra stress tensor respectively, p is the pressure, I is the identity tensor, λ_1 is the ratio of relaxation to retardation times, λ_2 is the retardation time, $\dot{\gamma}$ is shear rate, and a dot over the quantities indicates differentiation with respect to time.

Consider the steady, laminar, incompressible axisymmetric flow of two immiscible biofluids in an elastic tube of length L. The core region consists of Jeffrey fluid and the peripheral region consists of a Newtonian fluid. The radius of the undeformable tube is taken as a_0. Let the radii and viscosities of the core and peripheral regions be a_1, a_2 and μ_1, μ_2 respectively. The flow is axisymmetric. The cylindrical polar coordinates (r, θ, z) are chosen, where r and z denote the radial and axial coordinates and θ is the azimuthal angle. Let $u_1(r)$ and $u_2(r)$ be the fluid velocities in core and peripheral regions respectively.

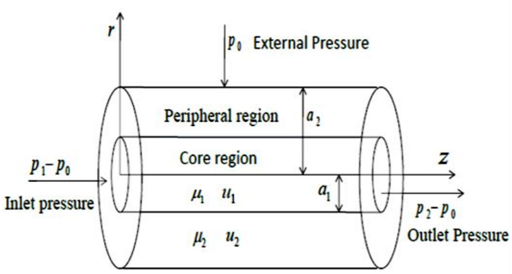

Fig. 1. Physical Model.

The basic equations governing the motion are as follows:

Core region (Jeffrey fluid):

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\mu_1}{1+\lambda_1} r \frac{\partial u_1}{\partial r}\right] = \frac{\partial p}{\partial z}, \quad 0 \le r \le a_1 \tag{3}$$

Peripheral region (Newtonian fluid):

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\mu_2 r \frac{\partial u_2}{\partial r}\right] = \frac{\partial p}{\partial z}, \quad a_1 \le r \le a_2 \tag{4}$$

where p is the pressure.

The boundary conditions are

$$\begin{aligned} &\text{at } r = a_1 : u_1 = u_2,\ \tau_1 = \tau_2, \\ &\text{at } r = a_2(z):\ u_2 = 0, \\ &\text{at } r = a_2(z):\ \psi_2 = \frac{Q}{2}, \\ &\text{at } r = a_1:\ \psi_1 = \psi_2. \end{aligned} \tag{5}$$

3. Solution of the Problem

The following non-dimensional quantities are used to determine the solution for Eq. (3) and Eq. (4) considering the boundary conditions given in Eq. (5).

$$\begin{aligned} &\bar{r} = \frac{r}{a_0},\ \bar{u}_1 = \frac{u_1}{U}, \bar{u}_2 = \frac{u_2}{U},\ \bar{a}_1 = \frac{a_1}{a_0},\ \bar{a}_2 = \frac{a_2}{a_0}, \\ &\bar{p} = \frac{a_0^{\,2}}{L\mu_1 U} p, \bar{z} = \frac{z}{L}, \bar{Q} = \frac{Q}{a_0 U},\ \bar{\psi}_i = \frac{\psi_i}{a_0 U}, \\ &\bar{\tau}_1 = \frac{\tau_1}{\left(\mu_1 U / a_0\right)}, \bar{\tau}_2 = \frac{\tau_2}{\left(\mu_1 U / a_0\right)}, \\ &\bar{\mu} = \begin{cases} 1,\ 0 \le r \le a_1 \\ \mu\left(= \dfrac{\mu_2}{\mu_1}\right),\ a_1 \le r \le a_2(z) \end{cases} \end{aligned} \tag{6}$$

After non-dimensionalization Eq. (3) and Eq. (4) becomes (dropping the bars)

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{1+\lambda_1} r \frac{\partial u_1}{\partial r}\right] = \frac{\partial p}{\partial z}, \quad 0 \le r \le a_1 \tag{7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\mu r \frac{\partial u_2}{\partial r}\right] = \frac{\partial p}{\partial z}, \quad a_1 \le r \le a_2 \tag{8}$$

The corresponding boundary conditions are

$$\text{at } r = a_2(z) : u_2 = 0 \tag{9a}$$

$$\text{at } r = a_1 : u_1 = u_2,\ \tau_1 = \tau_2 \tag{9b}$$

$$\text{at } r = a_2(z) : \psi_2 = \frac{Q}{2} \tag{9c}$$

$$\text{at } r = a_1 :\ \psi_1 = \psi_2 \tag{9d}$$

After solving the Eqs. (7) and (8) with the help of boundary conditions in Eqs. (9a) and (9b), we get

$$u_1 = \frac{P}{4}\left[(1+\lambda_1)\left(a_1^{\,2} - r^2\right) + \frac{1}{\mu}\left(a_2^{\,2} - a_1^{\,2}\right)\right], \quad 0 \le r \le a_1 \tag{10}$$

$$u_2 = \frac{P}{4\mu}\left[a_2^{\,2} - r^2\right], a_1 \le r \le a_2 \tag{11}$$

Here $P = -\frac{dp}{dz}$ is the pressure gradient.

The volumetric flow rate in the core and peripheral regions are denoted by Q_1 and Q_2 respectively and are given as

$$Q_1 = \frac{Pa_2^4}{8}\left\{(1+\lambda_1)\left(\frac{a_1}{a_2}\right)^4 + \frac{2}{\mu}\left(\frac{a_1}{a_2}\right)^2\left[1-\left(\frac{a_1}{a_2}\right)^2\right]\right\} \quad (12)$$

$$Q_2 = \frac{Pa_2^4}{8\mu}\left[1-\left(\frac{a_1}{a_2}\right)^2\right]^2 \quad (13)$$

The combined flux Q is given by

$$Q = Q_1 + Q_2 \quad (14)$$

$$Q = \frac{Pa_2^4}{8\mu}\left[1+\left(\frac{a_1}{a_2}\right)^4\left(\mu(1+\lambda_1)-1\right)\right] \quad (15)$$

Using the expressions for τ_1 and τ_2, obtain the following relation between ratio of shear stresses and radii at the tube wall and interface of the two fluids.

$$\frac{\tau_1}{\tau_2} = \frac{a_1}{a_2} = \tau, \quad (0<\tau<1) \quad (16)$$

In view of the above relation Eq. (15) becomes

$$Q = \frac{Pa_2^4}{8\mu}\left[1+\tau^4\left(\mu(1+\lambda_1)-1\right)\right], 0<\tau<1 \quad (17)$$

Taking $\lambda_1 = 0$ in Eq. (17) one can get the total flux for two immiscible Newtonian fluids in the tube. The observed flux by taking $\mu = 1, \lambda_1 = 0$ and $a_1 = a_2$ is exactly coincide with the results of Hagen-Poiseuille flow in a circular tube by Kapur (1985).

The flux given in Eq. (17) varies with the radius of the tube due to the elastic nature of the tube wall. Hence it is assumed that the Poiseuille law holds for two immiscible fluid flow in an elastic tube which will be discussed in the next section.

3. THEORETICAL FLUX

In this section, the authors calculated the theoretical flux Q for the flow of two immiscible Jeffrey and Newtonian fluids flow in an elastic tube of variable radius $a_2(z)$ and length L. Since the fluid flow fallows Poiseuille law at each cross section, Q is related to the pressure gradient by the following relation:

$$Q = \sigma(p - p_0)P \quad (18)$$

where $P = -\frac{dp}{dz}$,

$\sigma(p - p_0) = Fa_2^4$ and

$$F = \frac{1}{8\mu}\left[1+\tau^4\left(\mu(1+\lambda_1)-1\right)\right] \quad (19)$$

The expression given in Eq. (19) is similar to that of expression derived by Vajravelu *et al.* (2011) in the case of Newtonian fluid.

Here σ is the conductivity of the tube which is a function of the pressure difference $p(z) - p_0$, p_0 is the pressure outside the tube.

Integrating Eq. (18) from $z = 0$ at the inlet pressure $p(0) = p_1$, we have

$$Qz = \int_{p(z)-p_0}^{p_1-p_0} \sigma(p')dp' \quad \text{where} \quad p' = p(z) - p_0 \quad (20)$$

For $z = 1$ and $p(1) = p_2$, Eq. (20) reduces to

$$Q = \int_{p_2-p_0}^{p_1-p_0} \sigma(p')dp' \quad (21)$$

Since the radius of the tube a_2 is a function of $p - p_0$, Eq. (21) can be written as

$$Q = F\int_{p_2-p_0}^{p_1-p_0} a_2^4 dp' \quad (22)$$

Eq. (22) can be solved by considering the form of the function $a_2(p - p_0)$. If the stress or tension $T(a_2)$ in the tube wall is known as a function of a_2, then $a_2(p - p_0)$ is found using the equilibrium condition

$$\frac{T(a_2)}{a_2} = p - p_0 \quad (23)$$

Now it is necessary to know how the radius of a tube varies with pressure. The static pressure – volume relation is determined by Roach and Burton (1959) which is converted in to a tension versus length curve. This relation is represented by the following equation using Rubinow and Keller model (1972)

$$T(a_2) = t_1(a_2 - 1) + t_2(a_2 - 1)^5 \quad (24)$$

where $t_1 = 13, t_2 = 300$

From Eqs. (23) and (24), we obtain

$$dp' = \left\{\frac{t_1}{a_2^2} + t_2(4a_2^3 - 15a_2^2 + 20a_2 - 10 + \frac{1}{a_2^2})\right\}da_2 \quad (25)$$

Substituting Eq. (25) in Eq. (22), we get

$$Q = F\int_{p_2-p_0}^{p_1-p_0} a_2^4\left\{\frac{t_1}{a_2^2} + t_2\left(\begin{matrix}4a_2^3 - 15a_2^2 + 20a_2 \\ -10 + \frac{1}{a_2^2}\end{matrix}\right)\right\}da_2 \quad (26)$$

For $\mu = 1, \lambda_1 = 0$ and $\tau = 1$ Eq. (26) reduces to

$$Q = \frac{1}{8}\int_{p_2-p_0}^{p_1-p_0} a_2^4\left\{\frac{t_1}{a_2^2} + t_2\left(\begin{matrix}4a_2^3 - 15a_2^2 + 20a_2 \\ -10 + \frac{1}{a_2^2}\end{matrix}\right)\right\}da_2 \quad (27)$$

This is same as the expression derived by Vajravelu *et al.* (2011) for the case of Newtonian fluid.

The flux for immiscible fluids can be calculated by integrating Eq. (26).

$$Q = F\left[g(a_{21}) - g(a_{22})\right] \tag{28}$$

Where

$$g(a_2) = t_1\frac{a_2^{\,3}}{3} + t_2\begin{pmatrix} 4\frac{a_2^{\,8}}{8} - 15\frac{a_2^{\,7}}{7} \\ +20\frac{a_2^{\,6}}{6} - 10\frac{a_2^{\,5}}{5} + \frac{a_2^{\,3}}{3} \end{pmatrix}$$

$$a_{21} = a_2(\mathrm{p}_1 - p_0) \tag{29}$$

$$a_{22} = a_2(\mathrm{p}_2 - p_0)$$

The expressions given in Eq. (29) is same as the expression derived by Rubinow and Keller (1972) for the case of Newtonian fluid (single fluid) in an elastic tube.

By using the boundary conditions (9c) and (9d), we get the stream functions as

$$\psi_1 = \frac{Q}{2}\left[\frac{\mu(1+\lambda_1)r^2\left(2a_1^{\,2} - r^2\right) + 2r^2\left(a_2^{\,2} - a_1^{\,2}\right)}{a_2^{\,4} + a_1^{\,4}\left(\mu(1+\lambda_1) - 1\right)}\right], \quad 0 \le r \le a_1 \tag{30}$$

$$\psi_2 = \frac{Q}{2}\left[\frac{\begin{Bmatrix}\mu(1+\lambda_1)a_1^{\,4} + 2a_1^{\,2}\left(a_2^{\,2} - a_1^{\,2}\right) \\ +\left(r^2 - a_1^{\,2}\right)\left(2a_2^{\,2} - r^2 - a_1^{\,2}\right)\end{Bmatrix}}{a_2^{\,4} + a_1^{\,4}\left(\mu(1+\lambda_1) - 1\right)}\right], \quad a_1 \le r \le a_2 \tag{31}$$

When $\lambda_1 = 0$, the above expressions are similar to the expressions obtained by Ramachandra Rao and Usha (1995) for the flow of two immiscible viscous fluids in a circular tube in the absence of peristalsis.

Now the new expressions for the velocities of two fluids in terms of elastic parameters are obtained as

$$u_1 = \frac{2Q\left[\mu(1+\lambda_1)\left(a_1^{\,2} - r^2\right) + \left(a_2^{\,2} - a_1^{\,2}\right)\right]}{a_2^{\,4} + a_1^{\,4}\left[\mu(1+\lambda_1) - 1\right]} \tag{32}$$

$$u_2 = \frac{2Q\left[a_2^{\,2} - r^2\right]}{a_2^{\,4} + a_1^{\,4}\left[\mu(1+\lambda_1) - 1\right]} \tag{33}$$

Let $u = u_0$ be the interface velocity at $r = a_1$ and is given by

$$u_0 = \frac{2Q\left[a_2^{\,2} - a_1^{\,2}\right]}{a_2^{\,4} + a_1^{\,4}\left[\mu(1+\lambda_1) - 1\right]} \tag{34}$$

5. Results and Discussion

The results of the present study have interesting applications in understanding the fluid mechanics of biofluids such as blood behaves a two-fluid model rather than a single fluid model. The two-layered model facilitates various combinations of choosing the core and peripheral fluids through the viscosity ratio of the fluids and the non-Newtonian Jeffrey parameter. Two significant features are observed, one is change in interface velocity and other elasticity of the tube wall. In order to assess the quantitative effects of the various parameters involved in the problem, the numerical computations are carried out by using the software's Matlab and Mathematica.

5.1 Variation of flux (Jeffrey –Newtonian fluids)

The effect of elastic parameters t_1 and t_2, ratio of viscosities μ, ratio of radii τ, Jeffrey parameter λ_1, inlet and outlet pressures on flux Q have been numerically calculated from the Eq. (28) and the results are presented graphically in Figs. 2(a) - 2(g). In the present study, the choice of parameters for elastic parameters is considered from Rubinow and Keller (1972). From Fig. 2(a) and 2(b), it is noticed that the flux increases with the increasing values of elastic parameters. Figure 2(c) shows that the flux decreases with the ratio of viscosities. From Figs. 2(d) and 2(e), it is evident that the flux increases with Jeffrey parameter and ratio of radii. Figures 2(f) - 2(g) shows the variation in flux for different inlet and outlet pressures. Flux increases with the deceasing values of outlet pressure which is shown in Fig 2(f). A reverse trend is observed in the case increasing values of inlet pressure which is evident from Fig. 2(g).

5.2 Variation of Flux (Two Newtonian Fluids)

By taking $\lambda_1 = 0$ in Eq. (28), we get the flux of two Newtonian fluids of different viscosities. The effect of above said parameters on the flux of two Newtonian fluids of different viscosities are computed numerically and illustrated through graphs from Figs. 3(a) - 3(f). The effect of above mentioned parameters on the flux of two Newtonian fluids are similar to that of Jeffrey-Newtonian case. It is also observed that the flux is more in the case of Jeffrey-Newtonian case.

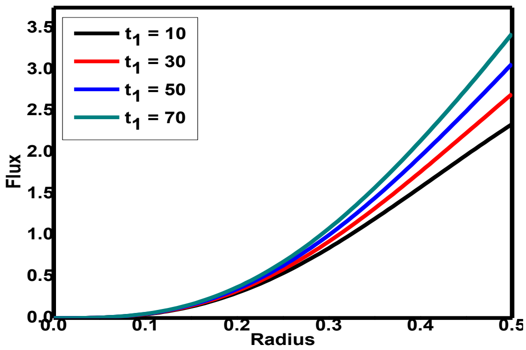

Fig. 2(a). Variation of Flux with Radius for $t_2 = 300, \mu = 0.3, \lambda_1 = 5, \tau = 0.5$.

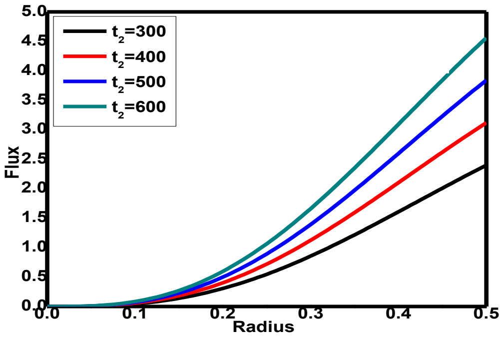

Fig. 2(b). Variation of Flux with Radius for $t_1 = 13, \mu = 0.3, \lambda_1 = 5, \tau = 0.5$.

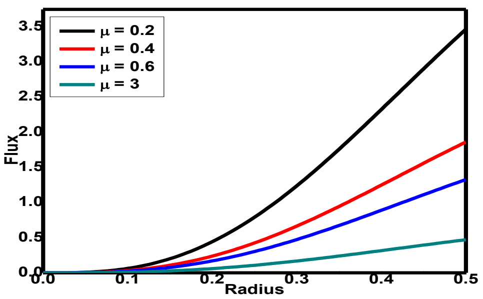

Fig. 2(c). Variation of Flux with Radius for $t_1 = 13, t_2 = 300, \lambda_1 = 5, \tau = 0.5$.

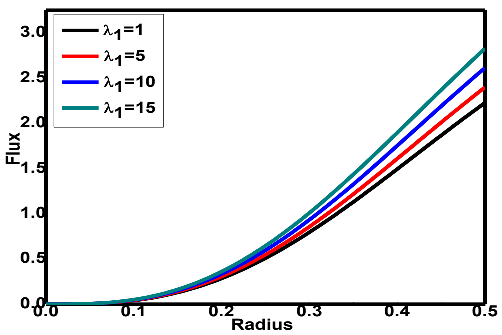

Fig. 2(d). Variation of Flux with Radius for $t_1 = 13, t_2 = 300, \mu = 0.3, \tau = 0.5$.

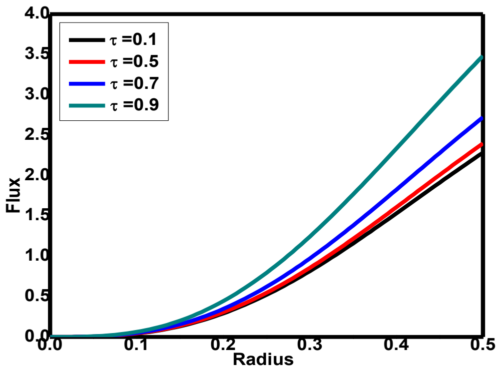

Fig. 2(e). Variation of Flux with Radius for $t_1 = 13, t_2 = 300, \mu = 0.3, \lambda_1 = 5$.

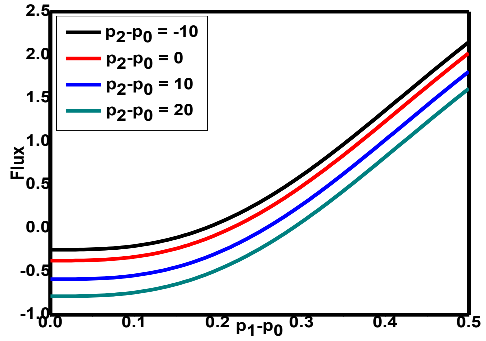

Fig. 2(f). Variation of Flux with inlet pressure for $t_1 = 13, t_2 = 300, \mu = 0.3, \lambda_1 = 5, \tau = 0.5$.

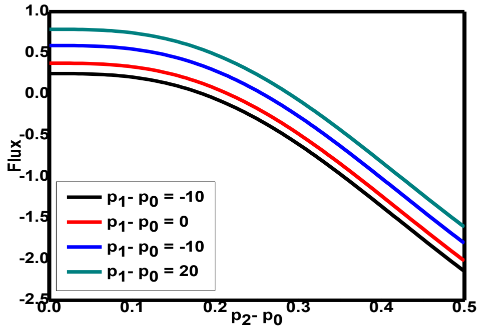

Fig. 2(g). Variation of Flux with outlet pressure for $t_1 = 13, t_2 = 300, \mu = 0.3, \lambda_1 = 5, \tau = 0.5$.

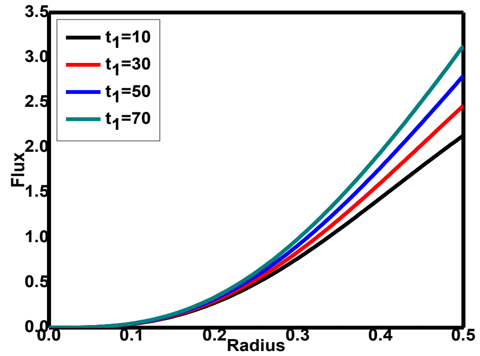

Fig. 3(a). Variation of Flux with Radius for $t_2 = 300, \mu = 0.3, \tau = 0.5$.

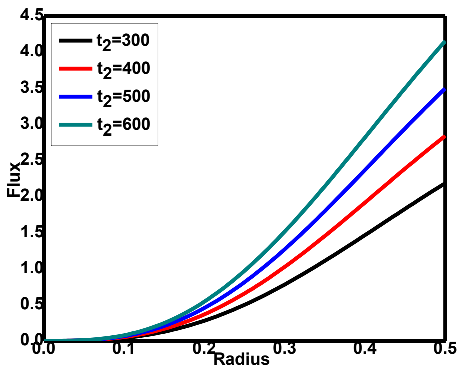

Fig. 3(b). Variation of Flux with Radius for $t_1 = 13, \mu = 0.3, \tau = 0.5$.

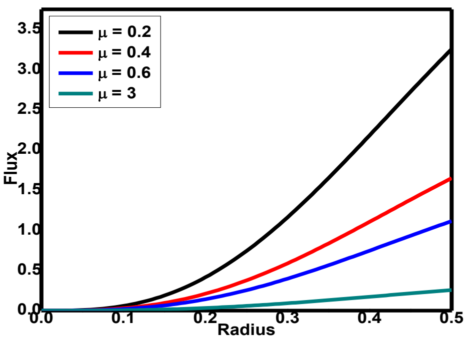

Fig. 3(c). Variation of Flux with Radius for $t_1 = 13, t_2 = 300, \tau = 0.5$.

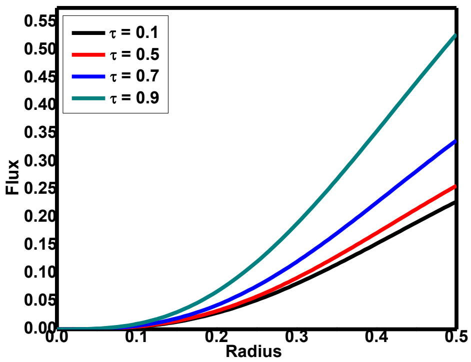

Fig. 3(d). Variation of Flux with Radius for $t_1 = 13, t_2 = 300, \mu = 0.3, \lambda_1 = 5$.

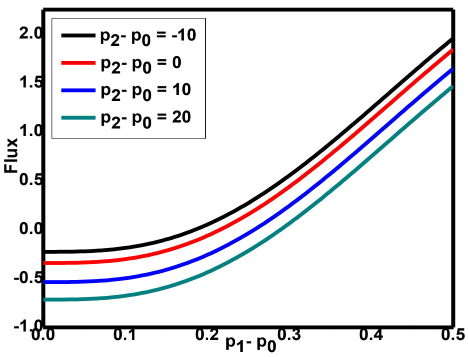

Fig. 3(e). Variation of Flux with inlet pressure for $t_1 = 13, t_2 = 300, \mu = 0.3, \tau = 0.5$.

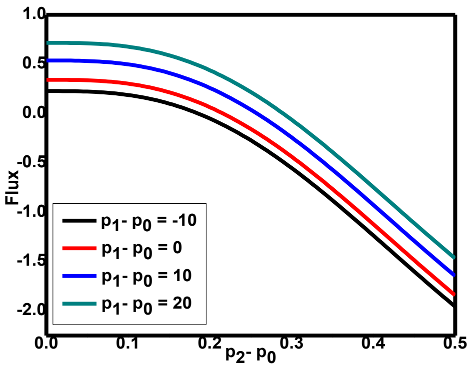

Fig. 3(f). Variation of Flux with outlet pressure for $t_1 = 13, t_2 = 300, \mu = 0.3, \tau = 0.5$.

5.3 Velocity Profiles (Jeffrey –Newtonian Fluids)

The effect of elastic parameters t_1 and t_2, ratio of viscosities μ, ratio of radii τ, Jeffrey parameter λ_1, on velocity of two fluids in core and peripheral regions at an inlet pressure $p_1 - p_0 = 10$ are evaluated from Eqs. (32) and (33) and are depicted in Figs. 4(a) - 4(e). In this case, the regions $0 \le r \le 0.125$ and $0.125 \le r \le 0.25$ represent the core and peripheral regions respectively. Velocity of the two layered fluid increases with increasing elastic parameters which is evident from Figs. 4(a) - 4(b) whereas opposite behaviour is observed in the case of increase of ratio of viscosities which is clear from Fig. 4(c). This may be due to the increase of viscosity of the peripheral layer. From Fig. 4(d), it is observed that velocity increases with Jeffrey parameter. Similar behaviour is noticed by Vajravelu *et al*. (2017) for two fluids flow in a channel under peristalsis. Since the peripheral region contains Newtonian fluid, there is no influence of λ_1 on velocity in the peripheral region.

5.4 Velocity profiles (Two Newtonian fluids)

By substituting $\lambda_1 = 0$ in Eqs. (32) and (33), we get the velocity of two Newtonian fluids and variation in velocity profiles regarding two Newtonian fluids for the parameters discussed in the above section 5.3 are displayed through Figs. 5(a) - 5(c). Same trends are observed as in the case of Jeffrey-Newtonian model and comparing the results with the previous section 5.3 higher velocities observed for Jeffrey-Newtonian fluids when compared with two Newtonian fluids.

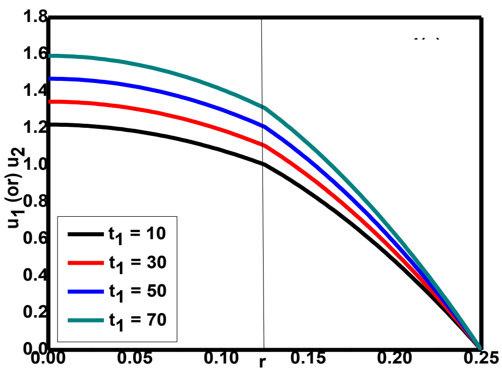

Fig. 4(a). Variation of Velocity with Radius for $t_2 = 300, \mu = 0.5, \lambda_1 = 0.3, \tau = 0.5$.

5.5 Shape of the Boundary of the Elastic Tube

In this section, the shape of the boundary of the elastic tube for different elastic parameters is discussed. From Eq. (18) and Eq. (23), we get a relation for the shape of tube in terms of a_2 and Z as follows:

$$t_1\left(a_2-1\right)+t_2\left(a_2-1\right)^5=a_2\left(50-\frac{Qz}{F}\left(\frac{\tau}{a_1}\right)^4\right)$$

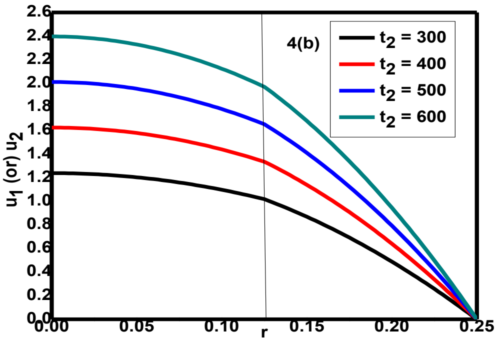

Fig. 4(b). Variation of Velocity with Radius for $t_1=13,\mu=0.5,\lambda_1=0.3,\tau=0.5$.

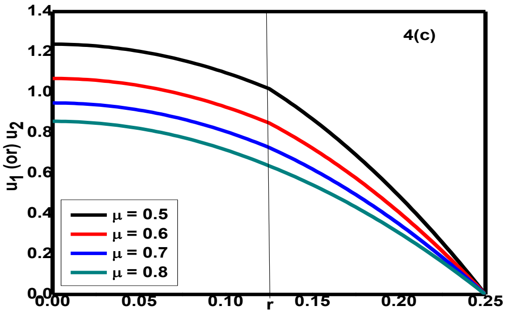

Fig. 4(c). Variation of Velocity with Radius for $t_1=13,t_2=300,\lambda_1=0.3,\tau=0.5$.

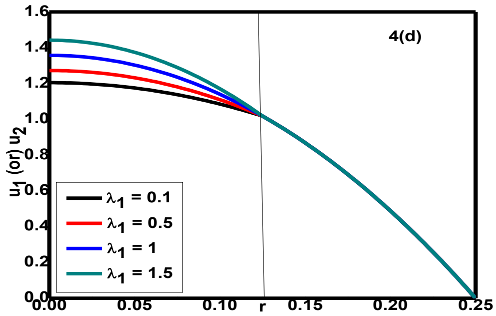

Fig. 4(d). Variation of Velocity with Radius for $t_1=13,t_2=300,\mu=0.5,\tau=0.5$.

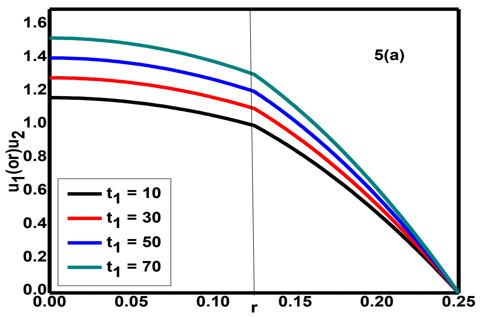

Fig. 5(a). Variation of Velocity with Radius for $t_2=300,\mu=0.5,\tau=0.5$.

From Figs. 6(a) - 6(b), as z increases from 0 to 1, it is seen that the boundary of tube is slanted towards the line $z=1$. This phenomenon is observed due to the elasticity in the walls of the tube. More variation is observed in the case of elastic parameter t_1.

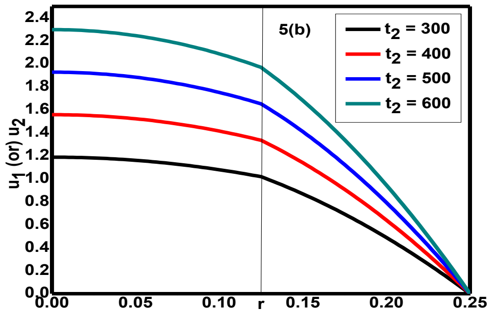

Fig. 5(b). Variation of Velocity with Radius for $t_1=13,\mu=0.5,\tau=0.5$.

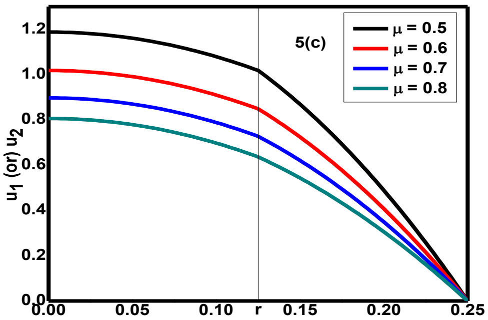

Fig. 5(c). Variation of Velocity with Radius for $t_1=13,t_2=300,\tau=0.5$.

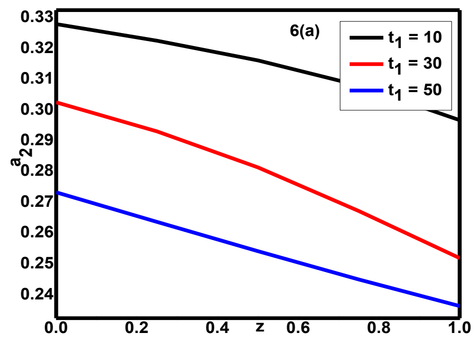

Fig. 6(a). Variation of radius of the tube a_2 with z for $t_2=300,\mu=0.5,\tau=0.5$.

5.6 Velocity Profiles at the Interface

To the best of literature survey, so far no discussion is made on the interface velocity of two immiscible fluids. The velocity profiles at the interface are represented through the Fig. 7(a) - 7(c). Interface velocity variation along elastic parameters t_1 and t_2 for different values of Jeffrey parameter λ_1 are shown in Figs. 7(a) and 7(b) respectively. It is observed that the interface velocity increases as Jeffrey parameter increases. Figure 7(c) illustrates the interface velocity variation along flow rate Q for different values of Jeffrey parameter λ_1. It is noticed that velocity at interface enhances as Jeffrey parameter increases. Because, when comparing from Newtonian case the increase in Jeffery parameter reduces the viscosity effect which causes increase in velocities in an elastic tube.

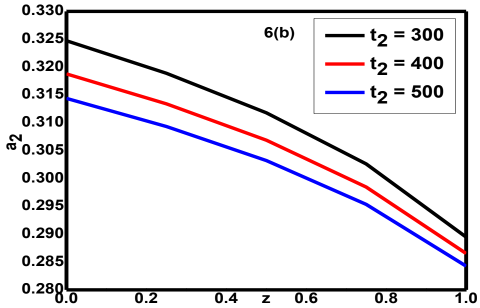

Fig. 6(b). Variation of radius of the tube a_2 with z for $t_1 = 13, \mu = 0.5, \tau = 0.5$.

5.7 Validation of the Present Work

Comparison result for flux of single phase Newtonian model (Vajravelu *et al.* (2011)) and two-phase Newtonian model (present work with $\lambda_1 = 0$) of fluid flow in an elastic tube have been displayed in Fig. 8. It is noticed that the flux is more in the case of two fluid model when compare to single fluid model.

Table I represents the comparison of flux for different values of t_1. The observed values are in good agreement with those of Vajravelu *et al.* (2011) and Rubinow and Keller (1972) for the flow of viscous fluid in an elastic tube.

6. Conclusions

The present work deals with the flow of a Jeffrey fluid in contact with a Newtonian fluid in an elastic tube. Analytical expressions for velocities of core and peripheral fluids along with flux flow rate are derived. Rubinow and Keller model was used to study the effects of various physical parameters on volume flow and velocity. The interesting facts are noticed as follows:

1. The flux increases for increasing values of elastic parameters, Jeffrey parameter and decreases with the values of ratio of viscosities for both the models i.e., Jeffrey Newtonian model and two Newtonian fluid model.
2. It is observed that the flux variation is more for Jeffrey Newtonian model when compare to two Newtonian fluids model.
3. Inlet and outlet pressures have opposite behaviour on the flux.
4. The velocity enhances with increasing values of elastic parameter, Jeffrey parameter and decreases with the values of ratio of viscosities. The same trends are observed in case of two Newtonian fluids model. It is noticed that there is no remarkable difference in the velocity variation for both the models.
5. The interface velocity increases with increasing the Jeffrey parameter in an elastic tube.

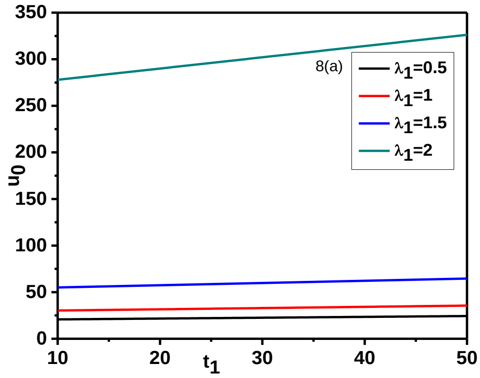

Fig. 7(a). Interface velocity u_0 with t_1 for fixed values of $t_2 = 300, \mu = 0.3, \tau = 0.5$.

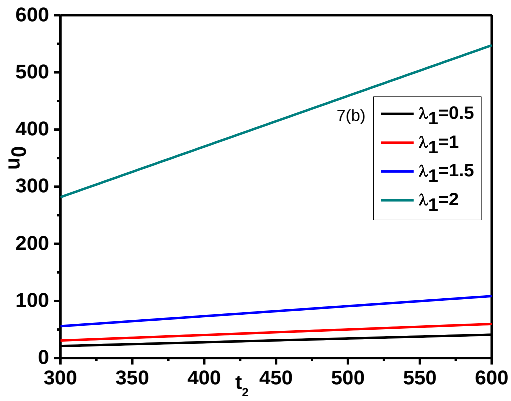

Fig. 7(b). Interface velocity u_0 with t_2 for fixed values of $t_1 = 13, \mu = 0.3, \tau = 0.5$.

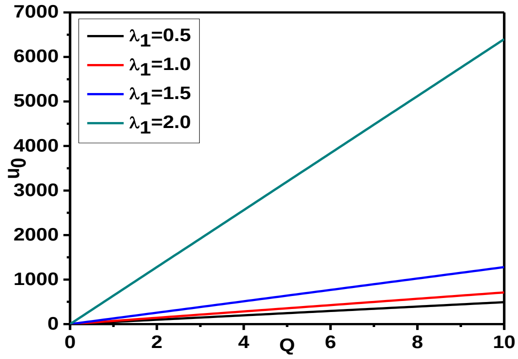

Fig. 7(c). Interface velocity u_0 with Q for fixed values of $t_1 = 13, t_2 = 300, \mu = 0.3, \tau = 0.5$.

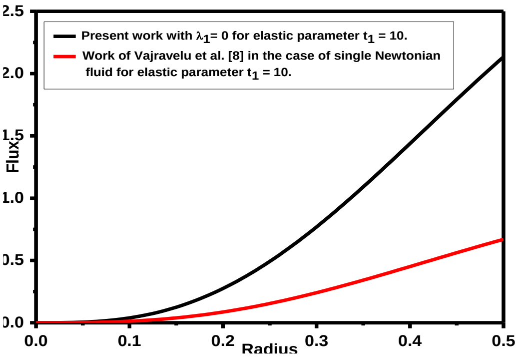

Fig. 8. Variation of Flux with Radius for $t_2 = 300, \mu = 0.3, \tau = 0.5$.

Table 1 Effect of elastic parameter t_1 on volumetric flow rate Q for fixed values of $t_2 = 300, \lambda_1 = 0, \mu = 1$ and $d = 1$

t_1	Q	Vajravelu *et al.* (2011) (for case of single Newtonian fluid)	Rubinow and Keller (1972)
10	0.154480	0.154480	0.154480
30	0.167501	0.167501	0.167501
50	0.180521	0.180521	0.180521

ACKNOWLEDGEMENTS

The authors express sincere thanks to the referees and the subject Editor for their valuable suggestions in the improvement of the paper.

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