



# Unsteady Unidirectional Flow of Voigt Fluid through the Parallel Microgap Plates with Wall Slip and Given Inlet Volume Flow Rate Variations

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(Received Nov 12, 2013; accepted April 13, 2014)

## ABSTRACT

In order to solve the velocity profile and pressure gradient of the unsteady unidirectional slip flow of Voigt fluid, Laplace transform method is adopted in this research. Between the parallel microgap plates, the flow motion is induced by a prescribed arbitrary inlet volume flow rate which varies with time. The velocity slip condition on the wall and the flow conditions are known. In this paper, two basic flow situations are solved, which are a suddenly started and a constant acceleration flow respectively. Based on the above solutions, linear acceleration and oscillatory flow are also considered.

**Keywords** Velocity Profile; pressure gradient; Voigt fluid; Laplace transform; Parallel microgap plates.

## NOMENCLATURE

$a_p$	constant acceleration	$u$	velocity in the x-coordinate direction
$\vec{b}$	body force field	$u_i$	tangential momentum of incoming molecules
$C_1, C_2$	coefficients	$u_p$	average inlet velocity
$F_v$	tangential momentum accommodation coefficient	$u_{re}$	tangential momentum of reflected molecules
$G$	rigidity modulus	$U_w$	tangential momentum
$h$	half of gap between two plates	$\vec{v}$	velocity vector
$i, j$	normal direction of plane	$x, y, z$	Cartesian coordinate system
$\vec{i}$	unit vector in the x-coordinate direction	$\delta_{ij}$	Kronecker delta
$p$	static fluid pressure	$\tau_{ij}$	shear stress
$Q$	given inlet volume flow rate	$\beta, \lambda$	velocity slip coefficient
$T_{ij}$	total stress	$\gamma$	shear strain
$\vec{T}$	total stress tensor	$\dot{\gamma}$	rate of shear strain
$t$	time	$\mu$	viscosity coefficient
$t'$	integration dummy variable	$\varepsilon$	slip parameter
$t_0$	time period of acceleration	$\Psi$	assumed particular solution
$U_p$	velocity after acceleration		

## 1. INTRODUCTION

Scientists and engineers are interested in the Newtonian fluid mechanics because it had been applied in a wide range of applications, such as chemical engineering, petroleum engineering, mechanical engineering, nuclear industries, geophysics and bioengineering. Unfortunately, very few liquids obey the laws of the Newtonian fluid. Therefore, the efforts needed for solving the non-Newtonian fluid is greatly increased. In real world applications, the behavior of blood, suspension fluids, certain oils and greases deviated from the laws of the Newtonian fluid. Some researchers classified the non-Newtonian fluids principally based on their behavior in shear stress and shear rate relation, for example, Maxwell fluid, Voigt fluid, Oldroyd-B fluid, Rivlin-Ericksen fluid and power-law fluid so on.

Due to rarefaction phenomena occurred in the microtube, the no-slip condition is no longer valid on the vicinity of the wall. Recently, Hayat *et al.* [1-7] conducted a series of studies which are related to the slip flow of non-Newtonian flow, including third grade fluid, Carreau fluid, Maxwell fluid, fourth grade fluid, second grade fluid, Burgers fluid and Johnson-Segalman fluid so on. Chen *et al.* [8-9] also studied the effect of rarefaction phenomenon on the flow pattern in the microtube. As the result of literatures review, the importance of slip flow problem is addressed. This is motivation for conducting this research.

Increasing efforts are being directed towards applying the technologies of Microfluidics to development of micro devices in engineering technologies. The no-slip condition is not enough to describe the behavior of microflow of Voigt fluid. As a Voigt fluid flows in microgap plates, the physical phenomenon of rarefaction will be considered and the typical flow field can be divided into four flow regimes by Knudsen number [14]: 1. Continuum flow as  $Kn < 10^{-3}$ ; 2. Slip flow as  $10^{-3} \leq Kn < 10^{-1}$ ; 3. Transition flow as  $10^{-1} \leq Kn < 10$ ; 4. Free molecular flow as  $Kn \geq 10$ . A few literature considered the influence of rarefaction in non-Newtonian fluid, and the paper studied the effects of rarefaction of an unsteady unidirectional flow between parallel microgap plates by Chen *et al.* [15] and Chen *et al.* [16].

## 2. MATHEMATICAL FORMULATIONS

The unidirectional rheological equation of state for a Voigt fluid in  $x$ -direction is given by Lee and Tsai [18]

$$T_{ij} = -p\delta_{ij} + \tau_{ij}, \quad i = x, y, z, \quad j = x \quad (1)$$

$$\tau_{yx} = G\gamma + \mu\dot{\gamma} \quad (2)$$

where  $T_{ij}$  is the total stress, subscript  $i$  denotes the normal direction of  $i$ -plane, subscript  $j$  denotes the stress acting direction,  $p$  is the static fluid pressure ( $p = p(x, y, z)$ ),  $\delta_{ij}$  is the Kronecker delta,  $\tau_{ij}$  is the shear stress,  $G$  is the rigidity modulus,  $\gamma$  is the shear strain,  $\dot{\gamma}$  is the rate of shear strain,  $\mu$  is the viscosity coefficient. Here  $G$ ,  $\mu$  are the material properties, and are assumed to be constant. When  $G=0$ , Eq. (2) reduces to that of a Newtonian fluid.

The problem of the unsteady flow of incompressible Voigt fluid between the parallel surfaces is considered. The dynamic equation is

$$\nabla \cdot \vec{T} + \rho \vec{b} = \rho \frac{d\vec{V}}{dt} \quad (3)$$

In the above equation,  $\nabla \cdot$  is the divergence operator,  $\vec{T}$  denotes the total stress tensor,  $\vec{b}$  is the body force field and  $\vec{V}$  is the velocity vector.

The continuity equation is

$$\nabla \cdot \vec{V} = 0 \quad (4)$$

Using the Cartesian coordinate system  $(x, y, z)$ , the  $x$ -axis is taken as the centerline direction between these two parallel surfaces,  $y$  is the coordinate normal to the plate,  $z$  is the coordinate normal to  $x$  and  $y$ , respectively, and the velocity field is assumed in the form

$$\vec{V} = u(y, t) \vec{i} \quad (5)$$

where  $u$  is the velocity in the  $x$ -coordinate direction,  $\vec{i}$  is the unit vector in the  $x$ -coordinate direction. This is effectively assumed that the flow is fully developed at all points in time.

Substituting of Eq. (5) into Eq. (4) shows that the continuity equation is automatically satisfied the result of substituting in Eq. (1) and Eq. (3). So we have the following scalar equation

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} - \rho \frac{\partial u}{\partial t} \quad (6)$$

where subscript  $y$  denotes the plane normal to  $y$  direction,  $x$  is the direction of the shear stress and

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \quad (7)$$

where the body force is incorporated into the term of the pressure gradient. Eq. (7) imply that the pressure gradient is a function of time and  $x$  only. Because the right hand side of Eq. (6) is not a function of  $x$ , the pressure gradient is a function of time only.

Solving Eq. (2) subject to  $\tau_{yx} = 0$  and  $\dot{\gamma} = \partial u / \partial y = 0$  as  $t = 0$ , the strain function is obtained

$$\gamma(t) = \frac{1}{\mu} e^{-\frac{G}{\mu}t} \int_0^t \tau_{yx} e^{\frac{G}{\mu}t'} dt' \quad (8)$$

where  $t'$  is the integration dummy variable. Eq. (6) and Eq. (8) are our governing equations describing the Voigt fluid flowing between the parallel surfaces.

## 3. METHODOLOGY OF SOLUTION

Since the governing equation, boundary conditions and initial condition are given, the problem is well defined. In general, it is not an easy work to solve this equation by the method of separation of variables or eigenfunction expansion. In this study, the Laplace transform method is used to treating the original partial differential equation. As the two variables reduce to one variable, the original partial differential equation becomes ordinary differential equation.

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} - \rho \frac{\partial u}{\partial t} \quad (9)$$

$$\gamma(t) = \frac{1}{\mu} e^{-\frac{G}{\mu}t} \int_0^t \tau_{yx} e^{\frac{G}{\mu}t'} dt' \quad (10)$$

As these two plates are  $2h$  apart, the wall slip and core flow condition are

$$u(h,t) = -\beta_v \lambda \frac{\partial u(h,t)}{\partial y} \quad (11)$$

where  $\beta_v \lambda$  is the velocity slip coefficient and is defined as

$$\beta_v \lambda = \frac{2 - F_v}{F_v} \quad (12)$$

and  $F_v$  is the tangential momentum accommodation coefficient that describes the interaction between fluid and wall and is related to constituents of fluid, temperature, velocity, wall temperature, roughness, and chemical status.  $F_v$  is defined as

$$F_v \equiv \frac{u_i - u_{re}}{u_i - U_w} \quad (13)$$

where  $u_i$ ,  $u_{re}$  and  $U_w$  are tangential momentum of incoming molecules, reflected molecules and re-emitted molecules respectively.

And

$$\frac{\partial u(0,t)}{\partial y} = 0 \quad (14)$$

The initial condition is related to inlet volume flow rate by

$$\int_{-h}^h u(y,t) dy = u_p(t) 2h = Q(t) \quad (15)$$

where  $u_p(t)$  is the known average inlet velocity and  $Q(t)$  is the given inlet volume flow rate function.

The above governing equations, boundary conditions and initial condition are prescribed and can be solved by the Laplace transform technique.

Differentiating Eq. (10) with respect to time and taking Laplace transform, then we have

$$\tau_{yx}(y,s) = \frac{\mu s + G}{s} \frac{du(y,s)}{dy} \quad (16)$$

Taking the Laplace transform of Eq. (9) and substituting in Eq. (16), then we have the governing equation

$$\frac{d^2 u(y,s)}{dy^2} - \frac{\rho s^2}{\mu s + G} u(y,s) = \frac{s}{\mu s + G} \frac{dP(x,s)}{dx} \quad (17)$$

Considering the governing equation as an ordinary differential equation with respect to  $y$  with boundary conditions

$$u(h,s) = \varepsilon \frac{du(h,s)}{dy}, \quad \varepsilon = -\beta_v \lambda \quad (18)$$

$$\frac{du(0,s)}{dy} = 0 \quad (19)$$

and the initial condition

$$\int_{-h}^h u(y,s) dy = u_p(s) 2h \quad (20)$$

The general solution of Eq. (17) is

$$u(y,s) = C_1 \sinh my + C_2 \cosh my + \Psi \quad (21)$$

where  $\Psi$  is the assumed particular solution and

$$m = \sqrt{\frac{\rho s^2}{\mu s + G}} \quad (22)$$

The boundary conditions Eq. (18) and Eq. (19) are used to solve the two arbitrary coefficients  $C_1$  and  $C_2$ .

Substituting  $C_1$  and  $C_2$  in Eq. (21), we have

$$u(y,s) = \Psi \left( 1 - \frac{\cosh my}{\cosh mh - \varepsilon m \sinh mh} \right) \quad (23)$$

Substituting Eq. (23) in the initial condition of Eq. (20),  $\Psi$  is readily obtained as

$$\Psi \int_{-h}^h \left( 1 - \frac{\cosh my}{\cosh mh - \varepsilon m \sinh mh} \right) dy = u_p(s) 2h \quad (24)$$

or

$$\Psi = \frac{u_p(s)(\cosh mh - \varepsilon m \sinh mh)}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \quad (25)$$

Substituting  $\Psi$  in Eq. (23) gives

$$\Psi(y,s) = \frac{u_p(s)(\cosh mh - \varepsilon m \sinh mh - \cosh my)}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \quad (26)$$

or

$$u(y,s) = u_p(s) \cdot \Omega(y,s) \quad (27)$$

where

$$\Omega(y,s) = \frac{\cosh mh - \varepsilon m \sinh mh - \cosh my}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \quad (28)$$

Taking the inverse Laplace transform, the velocity profile is

$$u(y,t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} u_p(s) \Omega(y,s) e^{st} ds \quad (29)$$

Furthermore, the pressure gradient is found by substituting Eq. (21) in Eq. (17) to give

$$\frac{dP(x,s)}{dx} = -\rho s \Psi \quad (30)$$

or

$$\frac{dP(x,s)}{dx} = -\rho s \frac{u_p(s)(\cosh mh - \varepsilon m \sinh mh)}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \quad (31)$$

Using the inverse transform formula, the pressure gradient distribution also can be obtained.

#### 4. ILLUSTRATION OF EXAMPLES

Hereafter, we will solve the cases proposed by Das and Arakeri [19] with the Voigt fluid to understand the different flow characteristics between these two fluids under the same condition.

For the first case, the piston velocity  $u_p(t)$  moves with a constant acceleration, and the second one, the piston

starts suddenly from rest and then maintains this velocity. Finally, the linear acceleration and oscillatory piston motion are also considered.

#### 4.1 Constant Acceleration Piston Motion

The piston motion of linear acceleration can be described by the following equation.

$$u_p(t) = a_p t = \left(\frac{U_p}{t_0}\right)t \quad (32)$$

where  $a_p$  is the constant acceleration,  $U_p$  is the final velocity after acceleration, and  $t_0$  is the time period of acceleration.

Taking the Laplace transform of Eq. (32) is

$$u_p(s) = \frac{U_p}{t_0 s^2} \quad (33)$$

From Eq. (27) and Eq. (33), the velocity profile is

$$u(y,t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \left( \frac{\frac{U_p}{t_0 s^2} \times e^{st} \times \left( \frac{\cosh mh - \varepsilon m \sinh mh - \cosh my}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \right)}{ds} \right) \quad (34)$$

From the above expression, the integration is determined using complex variable theory, as discussed by Arpacı [20]. It is easily observed that is a pole of order 2. Therefore, the residue at  $s = 0$  is

$$Res(0) = \frac{U_p}{t_0} \left\{ \frac{3t}{2h(3\varepsilon - h)} [(y^2 - h^2) + 2\varepsilon h] + \frac{\rho}{8\mu h(3\varepsilon - h)} y^4 + \frac{3\rho h(h - 5\varepsilon)}{20\mu(3\varepsilon - h)^2} y^2 - \frac{\rho h^3(h - 7\varepsilon)}{40\mu(3\varepsilon - h)^2} \right\} \quad (35)$$

The other singular points are the roots of following transcendental equation

$$\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh} = 0 \quad (36)$$

Setting  $m = \alpha i$ ,  $i = \sqrt{-1}$

$$\alpha h(\cos \alpha h + \varepsilon \alpha \sin \alpha h) - \sin \alpha h = 0 \quad (37)$$

If  $\alpha_n$ ,  $n = 1, 2, 3, \dots, \infty$  are roots of Eq. (37), then

$$s_{1n} = \frac{-\alpha_n^2 \mu + \alpha_n \sqrt{\alpha_n^2 \mu^2 - 4\rho G}}{2\rho}, n = 1, 2, 3, \dots, \infty \quad (38)$$

and

$$s_{2n} = \frac{-\alpha_n^2 \mu - \alpha_n \sqrt{\alpha_n^2 \mu^2 - 4\rho G}}{2\rho}, n = 1, 2, 3, \dots, \infty \quad (39)$$

are poles. These are simple poles, and residuals at all of these poles can be obtained as

$$Res(s_{1n}) = \frac{U_p \alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h) e^{s_{1n} t}}{t_0 s_{1n}^2 Q'_{1n}} \quad (40)$$

$$Res(s_{2n}) = \frac{U_p \alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h) e^{s_{2n} t}}{t_0 s_{2n}^2 Q'_{2n}} \quad (41)$$

where

$$\alpha'_{in} = -\frac{1}{2\alpha_n} \frac{\rho s_{in} (\mu s_{in} + 2G)}{(\mu s_{in} + G)^2}, i = 1, 2$$

$$Q'_{in} = \alpha h [\alpha'_{in} (\varepsilon - h) \sin \alpha_n h - \alpha'_{in} h \cos \alpha h + \varepsilon (\sin \alpha_n h + \alpha_n \alpha'_{in} h \cos \alpha_n h)], i = 1, 2 \quad (42)$$

Adding  $Res(0)$ ,  $Res(s_{1n})$  and  $Res(s_{2n})$ , a complete solution for the constant acceleration case is obtained as

$$\frac{u(y,t)}{U_p} = \frac{1}{t_0} \left\{ \frac{3t}{2h(3\varepsilon - h)} [(y^2 - h^2) + 2\varepsilon h] + \frac{\rho}{8\mu h(3\varepsilon - h)} y^4 + \frac{3\rho h(h - 5\varepsilon)}{20\mu(3\varepsilon - h)^2} y^2 - \frac{\rho h^3(h - 7\varepsilon)}{40\mu(3\varepsilon - h)^2} + \sum_{n=1}^{\infty} \left[ \frac{\alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h)}{s_{1n} Q'_{1n} + s_{2n} Q'_{2n}} \right] \right\} \quad (43)$$

where  $s_{1n}$ ,  $s_{2n}$ ,  $Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

The first term on the right-hand side of the above equation represents the steady-state velocity, the second term the transient response of the flow to an abrupt change either in the boundary conditions, body forces, pressure gradient or other external driving force.

Eq. (31) is used to determine the pressure gradient in this flow field, and follows the same procedure for solving velocity profile

$$Res(0) = \frac{U_p}{t_0} \left\{ \frac{3}{2h(3\varepsilon - h)} (2\mu t + Gt^2) + \frac{9\rho h(h - 5\varepsilon)}{5(3\varepsilon - h)^2} - \frac{3\rho(4\varepsilon - h)}{2(3\varepsilon - h)} \right\} \quad (44)$$

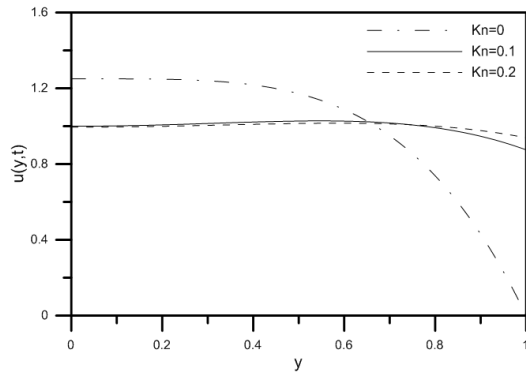
and

$$Res(s_n) = -\frac{\rho U_p}{t_0} \alpha_n h (\cos \alpha_n h + \varepsilon \alpha_n \sin \alpha_n h) \times \left( \frac{e^{s_{1n} t}}{s_{1n} Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n} Q'_{2n}} \right) \quad (45)$$

Therefore, the pressure gradient is

$$\frac{dP(x,t)}{dx} = \frac{U_p}{t_0} \left\{ \frac{3}{2h(3\varepsilon - h)} (2\mu t + Gt^2) + \frac{9\rho h(h - 5\varepsilon)}{5(3\varepsilon - h)^2} - \frac{3\rho(4\varepsilon - h)}{2(3\varepsilon - h)} - \rho \sum_{n=1}^{\infty} \alpha_n h (\cos \alpha_n h + \varepsilon \alpha_n \sin \alpha_n h) \left( \frac{e^{s_{1n} t}}{s_{1n} Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n} Q'_{2n}} \right) \right\} \quad (46)$$

where  $s_{1n}$ ,  $s_{2n}$ ,  $Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).



**Fig. 1. Velocity distribution under various  $\varepsilon$**

Figure 1 shows the velocity distribution under three different values of Kn. As Kn = 0, it means there is no slip condition. For Kn = 0.1 and 0.2, due to the rarefaction, the slip condition is occurred on the wall. As Kn becomes larger, the velocity  $u$  at the boundary velocity slip get larger.

The effects of rarefaction on unsteady flow through the microgap plates for arbitrary volume flow rate are obvious. For the result, the effects of wall-slip become significant with increasing rarefaction Kn .

**4.2 Suddenly Started Flow**

For a suddenly started flow between the parallel surfaces

$$u_p = 0, \quad t \leq 0$$

$$= U_p, \quad t > 0 \tag{47}$$

where  $U_p$  is the constant velocity. In which case the velocity profile is

$$\frac{u(y,t)}{U_p} = \frac{3}{2h(3\varepsilon - h)} [y^2 - h^2 + 2\varepsilon h]$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{\alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h)}{s_{1n} Q'_{1n} + s_{2n} Q'_{2n}} \left( \frac{e^{s_{1n} t}}{s_{1n} Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n} Q'_{2n}} \right) \right] \tag{48}$$

where  $s_{1n}, s_{2n}, Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

The pressure gradient is

$$\frac{dP(x,t)}{dx} = U_p \left[ \frac{3}{h(3\varepsilon - h)} (\mu + Gt) \right.$$

$$\left. - \rho \sum_{n=1}^{\infty} \alpha_n h (\cos \alpha_n h + \varepsilon \alpha_n \sin \alpha_n h) \left( \frac{e^{s_{1n} t}}{s_{1n} Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n} Q'_{2n}} \right) \right] \tag{49}$$

where  $s_{1n}, s_{2n}, Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

**4.3 Linear Acceleration Piston Motion**

The piston motion of linear acceleration can be described by the following equation.

$$u_p(t) = a_p t^2 = \left( \frac{U_p}{t_0} \right) t^2 \tag{50}$$

where  $a_p$  is the constant acceleration,  $U_p$  is the final velocity after acceleration, and  $t_0$  is the time period of acceleration.

In which case the velocity profile is

$$\frac{u(y,t)}{U_p} t_0 = \frac{3t^2}{2h(3\varepsilon - h)} [(y^2 - h^2) + 2\varepsilon h]$$

$$+ t \left[ \frac{\rho}{4\mu h(3\varepsilon - h)} y^4 + \frac{3\rho h(h - 5\varepsilon)}{10\mu(3\varepsilon - h)^2} y^2 + \frac{\rho h^3(3h - 5\varepsilon)}{20\mu(3\varepsilon - h)^2} \right]$$

$$+ \frac{\rho}{240\mu^2 h(3\varepsilon - h)} y^6 + \frac{\rho h(h - 5\varepsilon)}{80\mu(3\varepsilon - h)^2} y^4$$

$$+ \frac{3y^2}{2(3\varepsilon - h)} \left[ \varepsilon(6ih^4 + 4jh^2) - \frac{6i}{7} h^5 - \frac{4j}{5} h^3 \right]$$

$$- \left( \frac{i}{7} h^6 + \frac{j}{5} h^4 + \frac{2k}{3} h^2 \right)$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{2\alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h)}{s_{1n}^3 Q'_{1n} + s_{2n}^3 Q'_{2n}} \left( \frac{e^{s_{1n} t}}{s_{1n}^3 Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n}^3 Q'_{2n}} \right) \right] \tag{51}$$

where  $s_{1n}, s_{2n}, Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

$$i = \frac{\rho}{240\mu^2 h(3\varepsilon - h)},$$

$$j = \frac{\rho h(h - 5\varepsilon)}{80\mu(3\varepsilon - h)^2}, \tag{52}$$

$$k = \frac{3}{2(3\varepsilon - h)} \left[ \varepsilon(6ih^4 + 4jh^2) - \frac{6i}{7} h^5 - \frac{4j}{5} h^3 \right]$$

and the pressure gradient is

$$\frac{dP(x,t)}{dx} = \frac{U_p}{t_0} \left\{ \frac{Gt^3 + 3\mu t^2}{h(3\varepsilon - h)} \right.$$

$$+ \left[ \frac{3\rho(h - 2\varepsilon)}{3\varepsilon - h} + \frac{\rho h(h - 5\varepsilon)}{5(3\varepsilon - h)^2} \right] t + 2\mu k - \frac{\rho^2 h^3 (h - \varepsilon)}{4\mu(3\varepsilon - h)^2}$$

$$\left. - 2\rho \sum_{n=1}^{\infty} \alpha_n h (\cos \alpha_n h + \varepsilon \alpha_n \sin \alpha_n h) \left( \frac{e^{s_{1n} t}}{s_{1n}^2 Q'_{1n}} + \frac{e^{s_{2n} t}}{s_{2n}^2 Q'_{2n}} \right) \right\} \tag{53}$$

where  $s_{1n}, s_{2n}, Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

**4.4 Oscillatory Piston Motion**

The oscillating piston motion starting from rest is considered. The piston motion is described as

$$u_p = 0, \quad t \leq 0$$

$$= U_0 \sin(\omega t), \quad t > 0 \tag{54}$$

Taking the Laplace transform of Eq. (54), then we have

$$u_p(s) = \frac{U_0 \omega}{s^2 + \omega^2} \tag{55}$$

Substituting Eq. (54) in Eq. (27) to solve the velocity. These poles are simple poles at  $s = \pm \omega i$  and the roots

$$\text{of } \cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh} = 0.$$

The velocity profile is

$$\frac{u(y,t)}{U_0} = \frac{i}{2} [e^{-\omega t} \Omega(y, -\omega i) - e^{\omega t} \Omega(y, \omega i)] + \omega \sum_{n=1}^{\infty} \left[ \frac{\alpha_n h (\cos \alpha_n h - \cos \alpha_n y + \varepsilon \alpha_n \sin \alpha_n h)}{(s_{1n}^2 + \omega^2) Q'_{1n}} + \frac{e^{s_{2n} t}}{(s_{2n}^2 + \omega^2) Q'_{2n}} \right] \quad (56)$$

where  $s_{1n}$ ,  $s_{2n}$ ,  $Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

And the pressure gradient is obtained as

$$\frac{dP(x,t)}{dx} = -\frac{\rho U_0 \omega}{2} [e^{\omega t} \Gamma(\omega i) - e^{-\omega t} \Gamma(-\omega i)] - \rho U_0 \omega \left[ \sum_{n=1}^{\infty} \left( \frac{\alpha_n h (\cos \alpha_n h + \varepsilon \alpha_n \sin \alpha_n h)}{(s_{1n}^2 + \omega^2) Q'_{1n}} + \frac{s_{2n} e^{s_{2n} t}}{(s_{2n}^2 + \omega^2) Q'_{2n}} \right) \right] \quad (57)$$

where

$$\Gamma(s) = \frac{\cosh mh - \varepsilon m \sinh mh}{\cosh mh - \varepsilon m \sinh mh - \frac{\sinh mh}{mh}} \quad (58)$$

where  $s_{1n}$ ,  $s_{2n}$ ,  $Q'_n$  are defined in Eq. (38), Eq. (39) and Eq. (42).

## 5. CONCLUSION

Voigt fluids flow within the parallel microgap plates with wall slip conditions are solved by Laplace transform method. Comparing to the solutions of Chen *et al.* [12], the analytical solutions are identical when the slip parameter  $\varepsilon = 0$ . This article, [21-22] of our work and [23-24] provide more insight into how slip condition affects the non-Newtonian fluid flow pattern.

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