Analytical Solutions of Time Periodic Electroosmotic Flow in a Semicircular Microchannel

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Abstract

The time periodic electroosmotic flow of Newtonian fluids through a semicircular microchannel is studied under the Debye–Hückel approximation. Analytical series of solutions are found, and they consist of a time-dependent oscillating part and a time-dependent generating or transient part. Some new physical phenomena are found. The electroosmotic flow driven by an alternating electric field is not periodic in time, but quasi-periodic. There is a phase shift between voltage and flow, which is only dependent on the frequency of external electric field.

1. INTRODUCTION

When an electric field is applied in a channel, the walls are charged.The migration of the ions in excess induces the motion of the bulk solution due to the viscous drag. This phenomenon provides an attractive means for manipulating liquids in microdevices, and it has been widely used in different microdevices for various applications, such as microfractionation (Effenhauser 1995; Raymond*et. al.* 1996), electrophoresis (Harrison*et. al.* 1992), and microspray generation system (Ramsey and Ramsey 1997).

Time periodic electroosmotic flow is also known as AC electroosmosis, and it is driven by an alternating electric field. It is very important for biotechnology and separation science. Recently, various studies analyzed the time periodic electroosmotic flow theory and modeling in different geometries. Dutta and Beskok(2001) were among the early researchers who analytically investigated the time periodic electroosmotic flow between two parallel plates, illustrating interesting similarities or dissimilarities with the Stokes second problem. Based on the method proposed by Dutta and Beskok (2001), manyresearchers studied time periodic electroosmotic flows through microchannels, and some new results are given. General solutions are developed by Xuan and Li (2005) for direct current and alternating current electroosmotic flows in microfluidic channels with arbitrary cross-sectional geometry and arbitrary distribution of wall charge. Jian and his colleagues investigated the flow behavior of time periodic electro-osmosis in a cylindrical microannulus (*Jian et. al.* 2010; Bao*et. al.* 2013).Recently, Bandopadhyay and Chakraborty (2013) addressed the implications of finite sizes of the ionic species on electroosmotic transport through in narrow confinements in the case of a counterion-only solution, and pointed out that the electroosmotic mobility is dependent on both the size of the channel and the size of the ions.

Unfortunately, due to the incorrect critical assumption of the form of velocity distribution, the results given in these researches are not correct, and some very important physical phenomena have not been found theoretically. In their researches, these authors believed that the velocity profile will be oscillatory, and they assumed that the frequency of the oscillation is same as that of the externally applied electric fields. It is true that the electroosmotic flows should really be generated by the applied time periodic electric fields, and the flows may be time periodic. But, as we know, there is a phase difference between phase voltage and phase current, the flow in the microchannel needs some time to start. In other words, there is a phase difference between the the applied electric fields and electroosmotic flows. On another hand, on the basis of the aforementioned "*assumption*", the obtained analytical solutions of velocities are represented as complex functions, which is unreasonable in physics. In fact, the phase shift between the applied electric field and the flow response has been proved by Nayak (2013), as well as some other researchers (Luo 2004).

The steady/unsteady electroosmotic flow in an infinitely extended cylindrical channel with diameters ranging from 10 to 100nm has been investigated by Nayak (2013), and a degree of phase shift between the velocity field and the applied electric field is found numerically. Using the backwards-Euler time stepping numerical method. Luo (2004) clarified the relationship between the changes in the axial-flow velocity and the intensity of the applied electric field. Erickson and Li (2003) developed the analytical solution for AC electroosmotic flow through a rectangular microchannel for the case of a sinusoidal applied electric field.

The aim of the present paper is to providean analytical solutions for the time periodic electro-osmotic flow of Newtonian fluids through a semicircular microchannel. By the analytical solutions, we can easily understand the hidden physics or chemistry during the whole process.

2. FORMULATION OF THE PROBLEM

The motion of ionized, incompressible Newtonian fluid with electroosmotic body forces is governed by the following Navier-Stokes equation:

$$\rho \frac{D\boldsymbol{V}}{Dt} = -\nabla P + \mu \nabla^2 \boldsymbol{V} + \rho_e \boldsymbol{E} \tag{1}$$

where$P$is the pressure, ρ is the fluid density, $\mu$is the dynamic viscosity, $\boldsymbol{V}$is a divergence-free velocity field, i.e., $\nabla \cdot \boldsymbol{V} = 0$subject to the nonslipboundary conditions onthe walls, $\boldsymbol{E} = f(t)$is the externally applied electric field, and$\rho_e$is theelectric chargedensity, which can be expressed by apotential distribution Ψ

$$\nabla^2 \Psi = -\frac{\rho_e}{\varepsilon}, \tag{2}$$

and

$$\rho_e = 2n_0 z_v sinh\left(\frac{z_v e\Psi}{k_b T}\right). \tag{3}$$

Here n_0 is the bulk electrolyte concentration of a binary electrolyte dissociating into cations and anions of valence z_v, $e$is an electron charge, $k_b$is the Boltzmann constant, and $T$is the absolute temperature.

Considering the Debye-Hückel approximation under the cylindrical coordinate system (r, θ, z), equation (2)is linearized to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial \Psi}{\partial \varphi^2} = \kappa^2 \Psi \tag{4}$$

where $\kappa^2 = 2n_0 z_v^2 e^2 n_0 / \varepsilon k_b T$ is the Debye-Hückel parameter and $1/k$ is the Debye length.

Because of the effect of electric field, the fluid in the capillary will flow along the axial direction. Neglecting the pressure gradient along the axis, the Cauchy momentum equation in cylindrical coordinate system with AC electric field can be expressed as

$$\rho \frac{\partial u}{\partial t} = \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi}\right] + \rho_e E_0 \cos(\omega t), \tag{5}$$

where$u = u(r, \varphi, t)$is the axial velocity, $t$is time, and $E(t) = \rho_e E_0 \cos(\omega t)$ is the AC electric field

withE_0being the magnitude and ω being the frequency of the unsteady external electric field **E**.
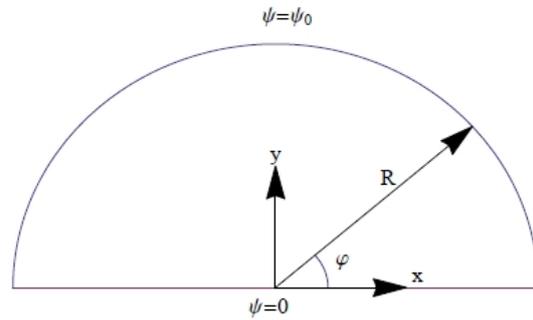
Fig. 1. The section of semicircular channel with different constant zeta potentials on the boundaries.

A semicircular microchannel is considered, as shown in Fig.1.

The radius of semicircular microchannelis R. The zeta potentials on the boundaries are assumed to vary. The potential is zero on the flat wall and Ψ_0 potential on the curved wall.

The following dimensionless variables are introduced

$$r^* = \frac{r}{R}, \Psi^* = \frac{\Psi}{\Psi_0}, u^*(r, \varphi) = \frac{u}{U_{eo}}, U_{eo} = -\frac{\varepsilon \Psi_0 E_0}{\mu}$$

For the dimensionless variables, we ignore the "*" notation for convenience. Then the governing equation for potential distribution

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = K^2 \Psi \tag{6}$$

$$\Psi = 0, \quad \varphi = 0, \tag{7}$$

$$\Psi = 0, \quad \varphi = \pi, \tag{8}$$

$$\Psi = 1, \quad r = 1, \tag{9}$$

and the equations for flow are

$$\frac{\partial u}{\partial t} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right] + K^2 \Psi \cos(\omega t) \tag{10}$$

$$u(r, \varphi, t) = 0, \quad \varphi = 0, \tag{11}$$

$$u(r, \varphi, t) = 0, \quad \varphi = \pi, \tag{12}$$

$$u(r, \varphi, t) = 0, \quad r = 1, \tag{13}$$

$$u(r, \varphi, t) = 0, \quad t = 0. \tag{14}$$

3. ANALYTICAL SOLUTIONS

We now consider the solution of the governing equations (6) and (10). Here it is not convenient to use the classical method of separation of variables because of the nonhomogeneities of the mast governing equation (10). It is for this reason that we consider the integral-transform technique, and this method provides a systematic, efficient, and straightforward approach for the solution of both homogeneous and nonhomogeneous, steady-state, and time dependent initial and boundary-value problems.

Introducing the following integral transform (*Ozisik*

1993) of the function $T(r,\varphi,t)$ with respect to the φ variable,

$$\overline{T}(r,n,t) = \int_0^\pi \sin(n\varphi)T(r,\varphi,t)d\varphi \ , \tag{15}$$

andits inversion

$$T(r,\varphi,t) = \sum_{n=1}^\infty \frac{2}{\pi}\sin(n\varphi)\overline{T}(r,n,t). \tag{16}$$

Here *n*are positive integers, and are the positive roots of $\sin n\pi = 0$.

The integral-transform pair in the *r*variable for the function $\overline{T}(r,n,t)$is defined as

$$\widetilde{\overline{T}}(\lambda_{v,m},n,t) = \int_0^1 rJ_v(\lambda_{v,m}r)\overline{T}(r,n,t)dr \tag{17}$$

$$\overline{T}(r,n,t) = \sum_{m=1}^\infty \frac{2J_v(\lambda_{v,m}r)}{J_v'^2(\lambda_{v,m}r)}\widetilde{\overline{T}}(\lambda_{v,m},n,t) \tag{18}$$

where$\lambda_{v,m}$are the positive roots of $J_v(\lambda_{v,m}r) = 0$, and $J_v(x)$ is Bessel function of the first kind.

Applying the above integral transform (15) and (16) to (6)-(13) yields

$$\frac{\partial^2 \overline{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial r} - \frac{n^2}{r^2}\overline{\Psi} = K^2\overline{\Psi} \ , \tag{19}$$

$$\overline{\Psi}(r,n) = \frac{1-(-1)^2}{n}, = 1 \ , \tag{20}$$
and

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2 \overline{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}}{\partial r} - \frac{n^2}{r^2}\overline{u} + K^2\overline{\Psi}\cos(\omega t) \tag{21}$$

$$\overline{u}(r,n,t) = 0, \ r = 1 \ , \tag{22}$$

$$\overline{u}(r,n,t) = 0 \ , \ t = 0 \ . \tag{23}$$

From (19) and (20) and with the help of (16), it is easy to get the solution for (6) in the following series form

$$\Psi(r,\varphi) = \sum_{n=1}^\infty \frac{2[1-(-1)^2]}{n\pi I_n(K)}I_n(Kr)\sin(n\varphi) \tag{24}$$

Applying integral transform (17) with respect to *r*to equation (21) results in

$$\frac{d\widetilde{\overline{u}}}{dt} = -\lambda_{n,m}^2\widetilde{\overline{u}} + K^2\cos(\omega t) \tag{25}$$

$$\widetilde{\overline{u}}(\lambda_{n,m},n,t) = 0 \text{for} t = 0 \tag{26}$$

This is an ordinary differential equation with initial condition, and its solution is

$$\widetilde{\overline{u}} = K^2\widetilde{\overline{\Psi}}\left[\sin(\omega t + \Phi_{n,m}) - \frac{\lambda_{n,m}^2}{\lambda_{n,m}^2+\omega^2}e^{-\lambda_{n,m}^2 t}\right] \tag{27}$$

where $\Phi_{n,m} = arctan(\omega/\lambda_{n,m}^2) < \pi/2$ is a phase difference or phase shift, and $\widetilde{\overline{\Psi}}(\lambda_{n,m},n)$ can be obtained from (2) with the help of the forementioned integral transforms (15) and (17). Then, substituting for $\widetilde{\overline{\Psi}}$ into equation (27) yields the distribution of velocity in the capillary,

$$u(r,\varphi,t) =$$
$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{8\sin[(2n-1)\varphi]J_{2n-1}(\lambda_{2n-1,m}r)K^2}{\pi\lambda_{2n-1,m}(2n-1)(\lambda_{2n-1,m}^2+K^2)J_{2n-1}'(\lambda_{2n-1,m})}$$
$$\times \left[\frac{\lambda_{2n-1,m}^2}{\lambda_{2n-1,m}^2+\omega^2}e^{-\lambda_{2n-1,m}^2 t} - \sin(\omega t + \Phi_{2n-1,m})\right] \tag{28}$$

4. RESULTS AND DISCUSSION

From the expression (28), we find that the velocity field of electroosmotic flow in the capillary generated by the external applied electric field is not time periodic in time. In particular, the distribution of velocity $u(r,\varphi,t)$can be written as the sum of a time-dependent oscillating part $u_1(r,\varphi,t)$ and a time-dependent generating part $u_2(r,\varphi,t)$,

$$u(r,\varphi,t) = u_1(r,\varphi,t) + u_2(r,\varphi,t), \tag{29}$$

where
$$u_1(r,\varphi,t) =$$
$$-\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{8\sin[(2n-1)\varphi]J_{2n-1}(\lambda_{2n-1,m}r)K^2}{\pi\lambda_{2n-1,m}(2n-1)(\lambda_{2n-1,m}^2+K^2)J_{2n-1}'(\lambda_{2n-1,m})} \tag{30}$$

$$\sin(\omega t + \Phi_{2n-1,m})$$
and

$$u_2(r,\varphi,t) =$$
$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{8\sin[(2n-1)\varphi]J_{2n-1}(\lambda_{2n-1,m}r)K^2}{\pi\lambda_{2n-1,m}(2n-1)(\lambda_{2n-1,m}^2+K^2)J_{2n-1}'(\lambda_{2n-1,m})}$$

$$\frac{\lambda_{2n-1,m}^2}{\lambda_{2n-1,m}^2+\omega^2}e^{-\lambda_{2n-1,m}^2 t} \tag{31}$$
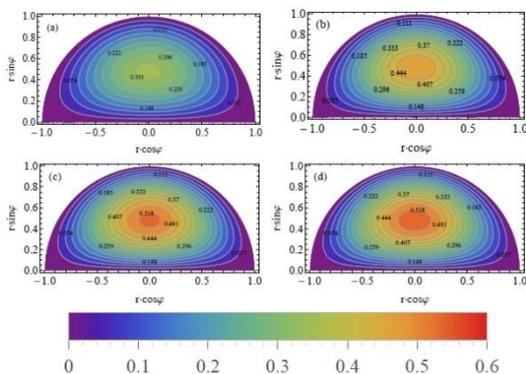
Fig. 2. Effect of *K* on the distribution of velocityprofile at given time *t* = 1:(a) *K* = 5, maximum velocity is 0.34634; (b) *K* = 10, maximumvelocity is 0.47936; (c) *K* = 20, maximumvelocity is 0.5302; (d) *K* = 50, maximumvelocity is 0.5465.
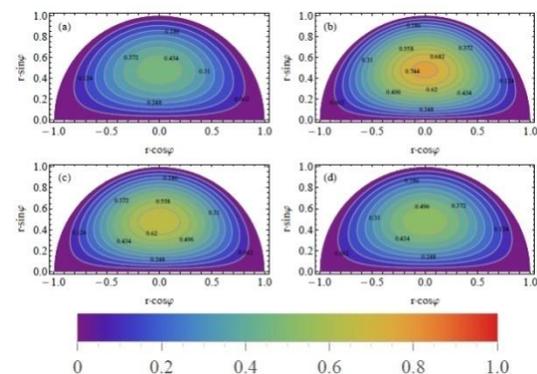
Fig. 3. The evolution of the velocity distribution in the tube for given ω = 20 and *K* = 10: (a) t= 1; (b) *t* = 8; (c) *t* = 16; (d) *t* = 24.

In fact, the minimum of $\lambda_{2n-1,m}$ is the first positive root of $J_1(\lambda_m) = 0$, i.e., $\lambda = min\{\lambda_{2n-1,m}\} \cong 3.8317$, and $e^{-a^2t} = 4.20456 \times 10^{-7}t$. It can be seen that the generating part of the solution (28) will tend to zero in a very short time. In this sense, the generating part can also be called a transient part. In other words, the electroosmotic flow generated by the AC electric field is quasi-periodic. The generating part of the solution is very important, since it explains both the characteristics of electroosmotic flow and the practical applications due to rapid development of the Biochiptechnology(Wong, Chen, Wang, and Ho 2004).

In particular, when $\omega \to 0$, $E(t) = E_0H(t)$ and

$$u(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[e^{-\lambda_m^2t} - 1\right]$$
$$\times \frac{8sin[(2n-1)\varphi]J_{2n-1}(\lambda_{2n-1,m}r)K^2}{\pi\lambda_{2n-1,m}(2n-1)(\lambda_{2n-1,m}^2+K^2)J'_{2n-1}(\lambda_{2n-1,m})} \quad (32)$$

Based on the aforementioned analysis of the generating part, the system of flow reaches a steady state instantaneously and finally we obtain

$$u(r,\varphi,t) = \sum_{n=1}^{\infty}\frac{4sin[(2n-1)\varphi]}{(2n-1)\pi}\left[r^{2n-1} - \frac{I_{2n-1}(Kr)}{I_{2n-1}(K)}\right] \quad (33)$$

Furthermore, let $\varphi = \phi + \pi/2$,

$$u(r,\phi,t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4sin[(2n-1)\phi]}{(2n-1)\pi}$$
$$\times \left[r^{2n-1} - \frac{I_{2n-1}(Kr)}{I_{2n-1}(K)}\right] \quad (34)$$

This is exactly the results given by Wang et al(Wang, Liu, and Chang 2008), which is on the basis of steady flow caused solely by electroosmosis.

The effects of K on the distribution of velocity inthe semi-capillary are shown in Fig.2. As K is increased, the velocity contours cease to be convex and the maximum velocity diffuses toward the boundary of the semi-capillary. Fig.3 presents the evolution of the velocity distribution in the tube for $\omega = 20$ and$K = 10$. As time elapses, it can be seen that the external AC electric filed causes the velocity of the fluid in the tube to oscillate. The maximum of velocity occurs in the center of the tube.

1. SUMMARY AND CONCLUSION

It has not been an accurate task to find the analytic solutions for time periodic electroosmotic flow (AC electroosmotic). In present research, we point out and correct the errors in the published research articles in this field, and we obtained an analytical solution for time-periodic electroosmotic flow in a semicircular microchannel. The velocity field of the electroosmosis flow consists of two parts, a time-dependent oscillating part and a time-dependent generating or transient part.

The transient part tends to zero very fast as time increases.The electroosmosis flow driven by an alternating electric field is not periodic in time, but quasiperiodic.There is a phase shift between voltage and flow, which is only dependent on the frequency of external electric field, and it is less than $\pi/2$.

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