



# Onset of Darcy-Brinkman Reaction-Convection in an Anisotropic Porous Layer

S. N. Gaikwad<sup>1†</sup> and M. Dhanraj<sup>2</sup>

<sup>1</sup> *Department of Mathematics, Gulbarga University, Kalaburagi - 585 106, Karnataka, India.*

<sup>2</sup> *Department of Mathematics, Lingaraj Appa Engineering College, Bidar – 585 403, Karnataka, India.*

†Corresponding Author Email: [sngaikwad2009@yahoo.co.in](mailto:sngaikwad2009@yahoo.co.in)

(Received February 28, 2014; accepted June 24, 2015)

## ABSTRACT

The linear and nonlinear stability analysis of double diffusive reaction-convection in a sparsely packed anisotropic porous layer subjected to chemical equilibrium on the boundaries is investigated analytically. The linear analysis is based on the usual normal mode method and the nonlinear theory on the truncated representation of Fourier series method. The Darcy-Brinkman model is employed for the momentum equation. The onset criterion for stationary, oscillatory and finite amplitude convection is derived analytically. The effect of Darcy number, Damkohler number, anisotropy parameters, Lewis number, and normalized porosity on the stationary, oscillatory, and finite amplitude convection is shown graphically. It is found that the effect of Darcy number and mechanical anisotropy parameter have destabilizing effect, while the thermal anisotropy parameter has stabilizing effect on the stationary, oscillatory and finite amplitude convection. The Damkohler number has destabilizing effect in the case of stationary mode, with stabilizing effect in the case of oscillatory and finite amplitude modes. Further, the transient behavior of the Nusselt and Sherwood numbers are investigated by solving the nonlinear system of ordinary differential equations numerically using the Runge-Kutta method.

**Keywords:** Double diffusive convection; Brinkman model; Chemical reaction; Anisotropy; Porous layer; Heat mass transfer.

## NOMENCLATURE

$a$	wavenumber, $\sqrt{l^2 + m^2}$	$Sh$	Sherwood number
$d$	height of the porous layer	$\Delta S$	concentration difference between the walls
$Da$	Darcy number, $\mu_e k_z / \mu d^2$	$T$	temperature
$\mathbf{g}$	gravitational acceleration, $(0, 0, -g)$	$\Delta T$	temperature difference between the walls
$\mathbf{K}$	permeability tensor, $K_x(\mathbf{ii} + \mathbf{jj}) + K_z(\mathbf{kk})$	$t$	time
$k$	lumped effective reaction rate	$x, y, z$	space coordinates
$Le$	Lewis number, $\kappa_{Tz} / \kappa_S$	$\beta_S$	solotal expansion coefficient
$l, m$	horizontal wavenumbers	$\beta_T$	thermal expansion coefficient
$Nu$	Nusselt number	$\varepsilon$	porosity
$p$	pressure	$\varepsilon_n$	normalized porosity parameter, $\varepsilon(\rho c)_f / (\rho c)_m$
$\mathbf{q}$	velocity vector, $(u, v, w)$	$\kappa_S$	solute diffusivity
$Ra_S$	solute Rayleigh number, $\rho_0 \beta_S g \Delta S d K_z / \mu \varepsilon \kappa_{Tz}$	$\kappa_T$	thermal diffusivity tensor, $\kappa_{Tx}(\mathbf{ii} + \mathbf{jj}) + \kappa_{Tz}(\mathbf{kk})$
$Ra_T$	thermal Rayleigh number, $\rho_0 \beta_T g \Delta T d K_z / \mu \varepsilon \kappa_{Tz}$	$\chi$	Damkohler number, $kd^2 / \varepsilon \kappa_{Tz}$
$S$	concentration	$\mu$	dynamic viscosity
$S_{eq}(T)$	equilibrium concentration of the solute at a given temperature	$\nu$	kinematic viscosity, $\mu / \rho_0$
		$\rho$	fluid density

$\xi$  mechanical anisotropy parameter,  
 $K_x/K_z$   
 $\eta$  thermal anisotropy parameter,  $\kappa_{T,x}/\kappa_{T,z}$   
 $\omega$  growth rate,  $\omega_r + i\omega_i$   
 $\psi$  stream function

**Other symbols**

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$$

**Subscripts**

$b$  basic state  
 $c$  critical  
 $f$  fluid phase  
 $0$  reference

**Superscripts**

$*$  dimensionless quantity  
 $'$  perturbed quantity  
 $F$  finite amplitude  
 $Osc$  oscillatory  
 $St$  stationary

**1. INTRODUCTION**

The study of double diffusive convection in a porous media is motivated both theoretically and by its practical applications in engineering. Some of the important areas of applications in engineering include the food and chemical process, solidification and centrifugal casting of metals, petroleum industry, and biomechanics and geophysical problems. Extensive reviews of the literature on this subject can be found in the books by Ingham and Pop (2005), Vafai (2000, 2005), Nield and Bejan (2006) and Vadasz (2008).

The linear stability analysis of thermohaline convection in a porous medium was performed firstly by Nield (1968) and found the criteria for the existence of steady and oscillatory thermohaline convection. Finite amplitude convection in a two-component fluid saturated porous layer has been studied by Rudraiah *et al.* (1982). It is found that a vertical solute gradient sets up over stable motions. The linear stability analysis of the thermosolutal convection is carried out by Poulikakos (1986) using the Darcy-Brinkman model. Some other researchers who have worked on double diffusive convection in a porous medium are Patil and Rudraiah (1980), Mamou (2002), Bahloul *et al.* (2003), Shivakumara *et al.* (2012) and Bhadauria *et al.* (2013).

Most of the studies are usually been concerned with homogeneous isotropic porous structures. In the last one decade, the effects of non-homogeneity and anisotropy of the porous medium have been studied. The geological and pedological processes rarely form isotropic medium as is usually assumed in transport studies. In a geothermal system with a ground structure composed of many strata of different permeabilities, the overall horizontal permeability may be up to ten times as large as the vertical component. Processes such as sedimentation, compaction, frost action, and reorientation of the solid matrix are responsible for the creation of anisotropic natural porous medium. Anisotropy can also be a characteristic of artificial porous materials such as pelleting used in chemical engineering processes, and fiber materials used in insulating purposes. Some of the studies related to anisotropic porous media are given by Tyvand (1980), Govender (2006), Malashetty and Swamy

(2010), Vanishree (2014) and Gaikwad and Kamble (2014).

Thermal convection is considered to be an important and in many practical cases a major mechanism for the transport and deposition of salts and other chemicals in sedimentary basins. A variety of chemical reactions can occur as fluid, carrying various dissolved species, moves through a permeable matrix. The nature of the resulting dissolution or precipitation depends on the reaction kinetics and the influence of temperature, pressure, and other factors on them has been studied by Phillips (2009). The effect of chemical reactions on convective motion is not fully known and has received relatively little attention. Influence of chemical reaction on double-diffusive convection in porous medium was first introduced by Steinberg and Brand (1983, 1984). Their analysis is restricted to the regime where the reaction rate was sufficiently fast that the solutal diffusion could be neglected. Gatica *et al.* (1989) and Viljoen *et al.* (1990) have examined the effect of exothermic-reaction on the stability of the porous system. Their study is limited to the case where the thermal and solutal diffusivities are equal so that overdamped oscillations are not possible. Linear stability analysis for chemically driven instabilities in binary liquid mixtures with fast chemical reaction was studied by Malashetty and Gaikwad (2003). They found analytical expressions for the onset of stationary and oscillatory instabilities. Pritchard and Richardson (2007) have been considered the effect of temperature dependent solubility on the onset of thermosolutal convection in an isotropic porous medium. Malashetty and Biradar (2011) have studied the onset of double diffusive reaction-convection in an anisotropic porous layer. Recently, Soret and Dufour effects on hydro magnetic heat and mass transfer over a vertical plate with a convective surface boundary condition and chemical reaction is performed by Gangadhar (2013).

It is well known that many applications in engineering disciplines as well as in circumstances linked to modern porous media involve high permeability porous media and in such situations the Darcy equation fails to give satisfactory results. So, use of non-Darcian models, which take care of boundary and/or inertia effects, is of fundamental

and practical interest to obtain accurate results for high permeability porous media. We shall apply Brinkman's model which has a Laplacian term analogous to that appearing in the momentum equation. Hence, non-Darcy effects on double diffusive convection in porous media have received a great deal of attention in the recent past. Givler and Altobelli (1994) have established that for high permeability porous media the effective viscosity is about ten times the fluid viscosity. Therefore, consideration of the ratio of effective viscosity to the fluid viscosity different from unity is warranted in order to know its influence on critical stability. Malashetty (1993) has studied the effects of anisotropic thermo convective currents on the double diffusive convection in a sparsely packed porous medium. A non-Darcy effect on double diffusive convection in a sparsely packed porous medium was studied by Shivakumara and Sumithra (1999). Kuznetsov and Nield (2010) analyzed the thermal instability in a porous layer saturated by a nanofluid using the Brinkman model. However, the studies on reaction-convection in a binary fluid saturated porous layer based upon the non-Darcian models are very sparse and it is in much-to-be desired state.

Therefore, in the present study we intend to perform linear and weak nonlinear stability analyses of double diffusive reaction-convection in a sparsely packed anisotropic porous layer. Our objective is to study how the onset criterion for stationary, oscillatory and finite amplitude convection is affected by the Darcy number, chemical reaction parameter and anisotropy parameters, and also to know their effect on heat and mass transfer in a more general porous medium.

## 2. MATHEMATICAL FORMULATION

We consider a sparsely packed, reactive anisotropic porous layer, saturated with Boussinesq fluid of infinite horizontal extent confined between the planes at  $z=0$  and  $z=d$ , with a distance  $d$  apart. A Cartesian frame of reference is chosen with the origin in the lower plane and the  $z$ -axis is vertical upward, where the gravity force  $\mathbf{g}$  is acting vertically downward. The lower plane is held at constant temperature  $T_0 + \Delta T$  and constant solute concentration  $S_0 + \Delta S$ , while the upper plane is held at  $T_0$  and  $S_0$  with  $\Delta T > 0$  and  $\Delta S > 0$ . The Darcy-Brinkman model is employed for the momentum equation and we are assuming that chemical equilibrium is maintained at the boundaries. The Boussinesq approximation is applied to account for the effects of density variations. With these assumptions the basic governing equations are

$$\nabla \cdot \mathbf{q} = 0, \tag{1}$$

$$\nabla p + \frac{\mu}{\mathbf{K}} \mathbf{q} - \mu_e \nabla^2 \mathbf{q} - \rho \mathbf{g} = 0, \tag{2}$$

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_f (\mathbf{q} \cdot \nabla) T = (\rho c)_m \nabla \cdot (\boldsymbol{\kappa}_T \cdot \nabla T) \tag{3}$$

$$\varepsilon \frac{\partial S}{\partial t} + (\mathbf{q} \cdot \nabla) S = \varepsilon \kappa_S \nabla^2 S + k(S_{eq}(T) - S) \tag{4}$$

$$\rho = \rho_0 [1 - \beta_T (T - T_0) + \beta_S (S - S_0)], \tag{5}$$

where the variables and constants have their usual meaning, as given in the Nomenclature. Furthermore, specific heat ratio  $\gamma = (\rho c)_m / (\rho c)_f$ , where  $(\rho c)_f$  is the volumetric heat capacity of the fluid and  $(\rho c)_m = (1 - \varepsilon)(\rho c)_s + \varepsilon(\rho c)_f$  is the volumetric heat capacity of the saturated medium as a whole, with the subscripts  $f, s$ , and  $m$  denoting the properties of the fluid, solid, and porous matrix, respectively. Following Pritchard and Richardson (2007), and Jupp and Woods (2003), it is assumed that the equilibrium solute concentration is a linear function of temperature so that  $S_{eq}(T) = S_0 + \phi(T - T_0)$ . Further, if chemical equilibrium at the boundaries is assumed, then  $\phi = \frac{S_0 + \Delta S - S_0}{T_0 + \Delta T - T_0} = \frac{\Delta S}{\Delta T}$ . The coefficient  $\phi$  in general may be positive or negative. Obviously, if  $\phi > 0$ , the solubility increases with temperature, while if  $\phi < 0$ , the solubility decreases with temperature. It should be noted that, only the case  $\phi > 0$  is considered in the present paper.

The boundary conditions are that at the upper boundary,  $T = T_0$ ,  $S = S_0$  and at the lower boundary  $T = T_0 + \Delta T$ ,  $S = S_0 + \Delta S$  and the vertical component of velocity vanishes at both the boundaries. The basic state of the fluid is assumed to be quiescent. We then find the temperature and solute distribution in the basic state as

$$T_b(z) = T_0 + \Delta T \left(1 - \frac{z}{d}\right) \text{ and } S_b(z) = S_0 + \Delta S \left(1 - \frac{z}{d}\right), \tag{6}$$

The initial distribution of solute is  $S_b = S_{eq}(T_b)$  and since  $S_{eq}$  is linear in  $T$ , we allow the existence of a steady basic state in which the solute is everywhere in chemical equilibrium with the solid matrix and therefore the vertical flux of solute is constant in space. We study the stability of this basic state using the method of small perturbations. Now superimpose the small perturbation at the basic state in the form

$$\mathbf{q} = \mathbf{q}_b + \mathbf{q}', \quad T = T_b(z) + T', \quad S = S_b(z) + S', \tag{7}$$

where the primes indicate perturbations. Substituting Eq. (7) into Eqs. (1)-(5), using the basic state solutions, we obtain equations governing the perturbations in the form

$$\nabla \cdot \mathbf{q}' = 0, \tag{8}$$

$$\nabla p' + \frac{\mu}{\mathbf{K}} \mathbf{q}' - \mu_e \nabla^2 \mathbf{q}' + \rho_0 (\beta_T T' - \beta_S S') \mathbf{g} = 0, \tag{9}$$

$$(\rho c)_m \frac{\partial T'}{\partial t} + (\rho c)_f \left( (\mathbf{q}' \cdot \nabla) T' - w' \frac{\Delta T}{d} \right) = (\rho c)_m \nabla \cdot (\kappa_T \nabla T'), \tag{10}$$

$$\varepsilon \frac{\partial S'}{\partial t} + (\mathbf{q}' \cdot \nabla) S' - w' \frac{\Delta S}{d} = \varepsilon \kappa_S \nabla^2 S' + k(S_{eq}(T') - S'). \tag{11}$$

By operating curl twice on Eq. (9), we eliminate  $p'$  and then use the scalings

$$(x, y, z) = (x^*, y^*, z^*)d, \quad t = \frac{d^2}{\kappa_{Tz}} t^*, \tag{12}$$

$$\mathbf{q}' = \frac{\varepsilon \kappa_{Tz}}{d} \mathbf{q}^*, \quad T' = (\Delta T) T^*, \quad S' = (\Delta S) S^*,$$

to obtain non-dimensionalize Eqs. (8)-(11) in the form (on dropping the asterisks for simplicity)

$$\left( \nabla_1^2 + \frac{1}{\xi} \frac{\partial^2}{\partial z^2} \right) w - Da \nabla^4 w - Ra_T \nabla_1^2 T + Ra_S \nabla_1^2 S = 0, \tag{13}$$

$$\frac{\partial T}{\partial t} + \varepsilon_n (\mathbf{q} \cdot \nabla) T - \varepsilon_n w = \eta \nabla_1^2 T + \frac{\partial^2 T}{\partial z^2}, \tag{14}$$

$$\frac{\partial S}{\partial t} + (\mathbf{q} \cdot \nabla) S - w = \frac{1}{Le} \nabla^2 S + \chi(T - S), \tag{15}$$

where the non-dimensional parameters are as defined in the Nomenclature. The boundary conditions for the dimensionless perturbation quantities are given by

$$w = T = S = 0 \text{ at } z = 0, 1. \tag{16}$$

### 3. LINEAR STABILITY ANALYSIS

In this section, we predict the thresholds of both stationary and oscillatory convection using linear theory. The eigen value problem defined by Eqs. (13)-(15) and subject to the boundary conditions (16) is solved using the time dependent periodic disturbances in a horizontal plane, upon assuming that amplitudes are small enough and can be expressed as

$$(w, T, S) = (w(z), \Theta(z), \Phi(z)) \exp[i(lx + my) + \omega t], \tag{17}$$

where  $l, m$  are the wavenumbers in the horizontal plane and  $\omega$  is the growth rate. Substituting Eq. (17) into Eqs. (13)-(15), we obtain

$$\left( \frac{D^2}{\xi} - a^2 - Da(D^2 - a^2)^2 \right) W + Ra_T a^2 \Theta - Ra_S a^2 \Phi = 0, \tag{18}$$

$$\left\{ \omega - (D^2 - \eta a^2) \right\} \Theta - \varepsilon_n W = 0, \tag{19}$$

$$\left\{ \omega - Le^{-1} (D^2 - a^2) + \chi \right\} \Phi - \chi \Theta - W = 0, \tag{20}$$

where  $D \equiv d/dz$  and  $a^2 = l^2 + m^2$ .

The boundary conditions (16) now read as

$$W = \Theta = \Phi = 0 \text{ at } z = 0, 1. \tag{21}$$

We assume the solutions to  $W, \Theta, \Phi$  in the form

$$(W(z), \Theta(z), \Phi(z)) = (W_0, \Theta_0, \Phi_0) \sin n\pi z, \tag{22}$$

$$(n = 1, 2, 3, \dots).$$

The most unstable mode corresponds to  $n=1$  which is the fundamental mode. Substituting Eq. (22) for the fundamental mode into Eqs. (18)-(20), and using the solvability condition we obtain an expression for thermal Rayleigh number as

$$Ra_T = \frac{(\delta_1^2 + Da \delta^4)(\omega + \delta_2^2)}{a^2 \varepsilon_n} + \left( \frac{\omega + \varepsilon_n \chi + \delta_2^2}{\omega + \chi + Le^{-1} \delta^2} \right) \frac{Ra_S}{\varepsilon_n}, \tag{23}$$

Where  $\delta^2 = \pi^2 + a^2$ ,  $\delta_1^2 = \pi^2 \xi^{-1} + a^2$ ,  $\delta_2^2 = \pi^2 + \eta a^2$ . The growth rate  $\omega$  is in general a complex quantity such that  $\omega = \omega_r + i\omega_i$ . The system with  $\omega_r < 0$  is always stable, while for  $\omega_r > 0$ , it will become unstable. For a neutral stability state  $\omega_r = 0$ .

### 3.1 Stationary Convection

For the validity of principle of exchange of stabilities (i.e., steady case), we have  $\omega = 0$  (i.e.,  $\omega_r = \omega_i = 0$ ) at the margin of stability. Therefore, for marginally stable steady mode Eq. (23) reads.

$$Ra_T^{St} = \frac{1}{a^2 \varepsilon_n} \left( a^2 + \pi^2 \xi^{-1} + Da(\pi^2 + a^2)^2 \right) \times \left( \pi^2 + \eta a^2 \right) + \frac{Le Ra_S}{\varepsilon_n} \left( \frac{\pi^2 + \eta a^2 + \varepsilon_n \chi}{Le \chi + \pi^2 + a^2} \right). \tag{24}$$

In the absence of Darcy number, i.e., when  $Da = 0$ , Eq. (24) reduces to

$$Ra_T^{St} = \frac{1}{a^2 \varepsilon_n} \left( a^2 + \frac{\pi^2}{\xi} \right) \left( \pi^2 + \eta a^2 \right) + \frac{Le Ra_S}{\varepsilon_n} \left( \frac{\pi^2 + \eta a^2 + \varepsilon_n \chi}{Le \chi + \pi^2 + a^2} \right). \tag{25}$$

This result coincides with the results of Malashetty and Biradar (2011).

### 3.2 Oscillatory Convection

For the marginally oscillatory state we now set  $\omega = i\omega_i$  in Eq. (23) and clear the complex quantities from the denominator, to obtain

$$Ra_T = \Delta_1 + i\omega_i \Delta_2, \tag{26}$$

$$\Delta_1 = \frac{(\delta_1^2 + Da\delta^4)\delta_2^2}{a^2\varepsilon_n} +$$

where

$$\left( \frac{\omega_i^2 + (\varepsilon_n\chi + \delta_2^2)(\chi + \delta^2Le^{-1})}{(\chi + \delta^2Le^{-1})^2 + \omega_i^2} \right) \frac{Ra_S}{\varepsilon_n},$$

$$\Delta_2 = \frac{(\delta_1^2 + Da\delta^4)}{a^2\varepsilon_n} +$$

$$\left( \frac{\chi(1 - \varepsilon_n) + \delta^2Le^{-1} - \delta_2^2}{(\chi + \delta^2Le^{-1})^2 + \omega_i^2} \right) \frac{Ra_S}{\varepsilon_n}.$$

Since  $Ra_T$  is a physical quantity, it must be real. Hence, from Eq. (26) it follows that either  $\omega_i = 0$  (principle of exchange of stabilities) or  $\Delta_2 = 0$  ( $\omega_i \neq 0$ , oscillatory onset).

For oscillatory convection  $\Delta_2 = 0$  ( $\omega_i \neq 0$ ) and this gives an expression for the frequency of oscillations in the form

$$\omega_i^2 = \frac{a^2Ra_S}{(\delta_1^2 + Da\delta^4)} (\delta_2^2 - \delta^2Le^{-1} -$$

$$\chi(1 - \varepsilon_n)) - (\chi + \delta^2Le^{-1})^2,$$

Now Eq. (26) with  $\Delta_2 = 0$ , gives

$$Ra_T^{Osc} = \frac{(\delta_1^2 + Da\delta^4)\delta_2^2}{a^2\varepsilon_n} + \left( \frac{\omega_i^2 + (\varepsilon_n\chi + \delta_2^2)(\chi + \delta^2Le^{-1})}{(\chi + \delta^2Le^{-1})^2 + \omega_i^2} \right) \frac{Ra_S}{\varepsilon_n}. \quad (28)$$

The analytical expression for the oscillatory Rayleigh number given by Eq. (28) is minimized with respect to the wavenumber numerically, after substituting for  $\omega_i^2$  ( $>0$ ) from Eq. (27), for various values of physical parameters in order to know their effect on the onset of oscillatory convection.

#### 4. NONLINEAR STABILITY ANALYSIS

In this section we consider the nonlinear analysis using a truncated representation of Fourier series containing only two terms. Although the linear stability analysis is sufficient for obtaining the stability condition of the motionless solution and the corresponding eigenfunctions describing qualitatively the convective flow, it cannot provide information about the values of the convection amplitudes and hence about the rate of heat and mass transfer. To obtain this additional information, we perform the nonlinear analysis, which is useful to understand the physical mechanism with a minimum amount of mathematics and it is a step forward toward understanding the full nonlinear problem.

For simplicity of analysis, we confine ourselves to two-dimensional rolls, so that all the physical quantities are independent of  $y$ . We introduce stream function  $\psi$  such that  $u = \partial\psi/\partial z$ ,  $w = -\partial\psi/\partial x$  into the Eq. (9), eliminate the pressure and non-dimensionalize the resulting

equation and Eqs. (10) and (11) using the transformations (12) to obtain

$$\left( \frac{1}{\xi} \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} - Da\nabla^4 \right) \psi + Ra_T \frac{\partial T}{\partial x} - Ra_S \frac{\partial S}{\partial x} = 0, \quad (29)$$

$$\left( \frac{\partial}{\partial t} - \eta \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) T - \varepsilon_n \frac{\partial(\psi, T)}{\partial(x, z)} + \varepsilon_n \frac{\partial\psi}{\partial x} = 0, \quad (30)$$

$$\left( \frac{\partial}{\partial t} - Le^{-1}\nabla^2 \right) S - \frac{\partial(\psi, S)}{\partial(x, z)} + \frac{\partial\psi}{\partial x} - \chi(T - S) = 0, \quad (31)$$

Now consider a minimal Fourier series with one term in the stream function, and to get some effects of nonlinearity, take two terms in the temperature and concentration fields as given below

$$\psi = A(t)\sin(ax)\sin(\pi z), \quad (32)$$

$$T = B(t)\cos(ax)\sin(\pi z) + C(t)\sin(2\pi z), \quad (33)$$

$$S = D(t)\cos(ax)\sin(\pi z) + E(t)\sin(2\pi z), \quad (34)$$

where the amplitudes  $A, B, C, D$ , and  $E$  are functions of time and are to be determined. Substituting Eqs. (32) - (34) in Eqs. (29) - (31) and equating the coefficients of like terms of the resulting equations, we obtain the following system of nonlinear differential equations:

$$\frac{dA}{dt} = -(\delta_1^2 + Da\delta^4)A - aRa_TB + aRa_SD, \quad (35)$$

$$\frac{dB}{dt} = -\delta_2^2B - a\varepsilon_nA - a\pi\varepsilon_nAC, \quad (36)$$

$$\frac{dC}{dt} = -4\pi^2C + \frac{\varepsilon_n\pi a}{2}AB, \quad (37)$$

$$\frac{dD}{dt} = -aA - Le^{-1}\delta^2D - \pi aAE + \chi(B - D), \quad (38)$$

$$\frac{dE}{dt} = -4\pi^2Le^{-1}E + \frac{\pi a}{2}AD + \chi(C - E). \quad (39)$$

The above nonlinear system of autonomous differential equations for time dependent variables is not suitable to be solved analytically, and thus it is to be solved using a numerical method. After determining the value of the amplitude functions  $A, B, C, D$  and  $E$ , we will obtain the expressions for the Nusselt number and Sherwood number as a function of time.

#### 4.1. Steady Finite Amplitude Motions

From qualitative predictions, we look into the possibility of an analytical solution. In the case of steady motions, Eqs. (35)-(39) can be solved in closed form. Setting the left hand side of Eqs. (35)-(39) equal to zero and elimination of all amplitudes, except  $A$ , gives

$$b_1 s^2 + b_2 s + b_3 = 0, \tag{40}$$

Where  $s = A^2/8$  and

$$\begin{aligned} b_1 &= 4a^4 Le^2 \pi^2 \varepsilon_n^2 (\delta_1^2 + Da \delta^4), \\ b_2 &= 4\pi^2 a^4 Le \varepsilon_n (\varepsilon_n Ra_S - Le Ra_T) + 4\pi^2 a^2 \\ &\quad \times (\delta_1^2 + Da \delta^4) (\delta^2 \varepsilon_n^2 + Le (\chi \varepsilon_n^2 + Le \delta_2^2)) \\ &\quad + a^2 \varepsilon_n^2 Le \chi (\delta_1^2 + Da \delta^4) (\delta^2 + Le \chi), \\ b_3 &= (4\pi^2 + \chi Le) \left\{ a^2 (Ra_S Le (\delta_2^2 + \chi \varepsilon_n) - Ra_T \varepsilon_n \right. \\ &\quad \left. \times (\delta^2 + Le \chi)) + (\delta_1^2 + Da \delta^4) (\delta^2 + Le \chi) \delta_2^2 \right\}. \end{aligned}$$

The required root of Eq. (40) is,

$$s = \frac{1}{2b_1} \left( -b_2 + (b_2^2 - 4b_1 b_3)^{1/2} \right). \tag{41}$$

When we let the radical in the above equation vanish, we obtain the expression for a finite amplitude Rayleigh number  $Ra_T^F$ , which characterizes the onset of finite amplitude steady motions:

$$Ra_T^F = \frac{1}{2c_1} \left( -c_2 + (c_2^2 - 4c_1 c_3)^{1/2} \right), \tag{42}$$

where the constants  $c_1, c_2$ , and  $c_3$  are not presented here for brevity.

#### 4.2. Heat and Mass Transport

In the study of convection in fluids, the quantification of heat and mass transport is important. This is because the onset of convection, as the Rayleigh number is increased, is more readily detected by its effect on the heat and mass transport. In the basic state, heat and mass transport is by conduction alone. We now proceed to find the Nusselt number and Sherwood number.

If  $H$  and  $J$  are the rate of heat and mass transport per unit area respectively, then

$$\begin{aligned} H &= -\kappa_{Tz} \left\langle \frac{\partial T_{total}}{\partial z} \right\rangle_{z=0} \quad \text{and} \\ J &= -\kappa_S \left\langle \frac{\partial S_{total}}{\partial z} \right\rangle_{z=0}, \end{aligned} \tag{43}$$

where the angular bracket corresponds to a horizontal average and

$$\begin{aligned} T_{total} &= T_0 - \Delta T \frac{z}{d} + T(x, z, t) \quad \text{and} \\ S_{total} &= S_0 - \Delta S \frac{z}{d} + S(x, z, t). \end{aligned} \tag{44}$$

Substituting Eqs. (33) and (34) into Eq. (44), and using the resultant equations into Eq. (43), we get

$$H = \frac{\kappa_{Tz} \Delta T}{d} (1 - 2\pi C) \quad \text{and} \quad J = \frac{\kappa_S \Delta S}{d} (1 - 2\pi E). \tag{45}$$

The Nusselt number and Sherwood number are defined respectively by

$$\begin{aligned} Nu &= \frac{H}{\kappa_{Tz} \Delta T/d} = 1 - 2\pi C \quad \text{and} \\ Sh &= \frac{J}{\kappa_S \Delta S/d} = 1 - 2\pi E \end{aligned} \tag{46}$$

Writing  $C$  and  $E$  in terms of  $A$  and substituting into Eq. (46), we obtain

$$\begin{aligned} Nu &= 1 + 2 \left( \frac{s}{s + \delta_2^2 / \varepsilon_n^2 a^2} \right), \tag{47} \\ Sh &= 1 + 2 \left( \frac{4\pi^2 a^2 Le^2 s (\delta_2^2 + a^2 \varepsilon_n^2 s + \chi \varepsilon_n) + a^2 Le \chi \varepsilon_n^2 s (\delta^2 + \chi Le)}{(4\pi^2 + \chi Le) (\delta^2 + \chi Le) + 4\pi^2 a^2 Le^2 s} \right) \\ &\quad \times (\delta_2^2 + a^2 \varepsilon_n^2 s) \end{aligned} \tag{48}$$

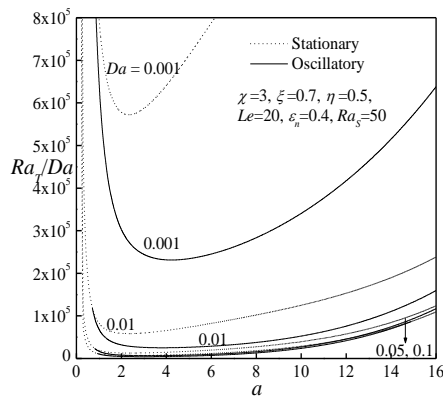
The second term on the right-hand side of Eqs. (47) and (48) represents the convective contribution to the heat and mass transport, respectively.

### 5. RESULTS AND DISCUSSION

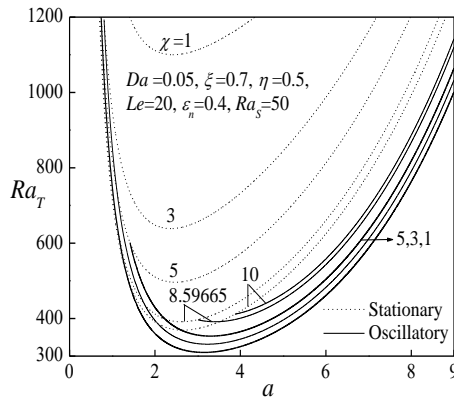
The onset of double diffusive reaction-convection in a sparsely packed anisotropic porous layer is studied analytically using the linear and nonlinear theories. The linear theory is based on the usual normal mode technique and the nonlinear theory on the truncated representation of the Fourier series method. The expressions for the stationary, oscillatory and finite amplitude Rayleigh numbers for different values of the parameters such as Darcy number, Damkohler number, mechanical anisotropy parameter, thermal anisotropy parameter, Lewis number, solute Rayleigh number, and normalized porosity are computed and their effects are shown through figures. We fixed the values of the parameters as  $Da = 0.05$ ,  $\chi = 3$ ,  $\xi = 0.7$ ,  $\eta = 0.5$ ,  $Le = 20$ ,  $Ra_S = 50$ , and  $\varepsilon_n = 0.4$  except the varying parameter.

Figure 1 shows the neutral stability curves for stationary and oscillatory modes for different values of Darcy number  $Da$  and for fixed values of other parameters. It is worth mentioning here that the Rayleigh number is normalized with respect to the Darcy number in the sense that the Rayleigh number is modified for the clear fluid. We observed from this figure that the minimum value of the Rayleigh number i.e.,  $Ra_{Tc}/Da$  for both stationary and oscillatory modes decreases with an increase in the value of the Darcy number  $Da$ , indicating that the effect of Darcy number is to destabilize the system. This is because, with increasing the Darcy number the permeability increases and thus the viscous drag decreases. Therefore, the critical Rayleigh number decreases. Moreover, the wavenumber at which the minimum of the Rayleigh

number for both stationary and oscillatory modes decreases with an increase of  $Da$ , indicating that the wavelength increases with  $Da$



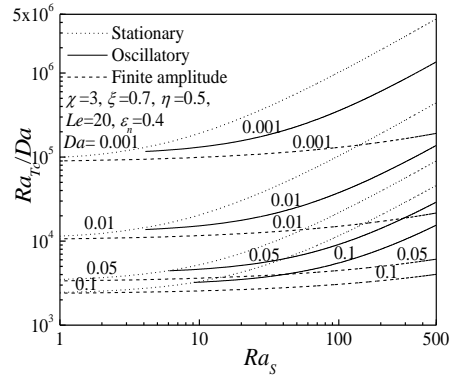
**Fig. 1. Neutral stability curves for different values of Darcy number  $Da$ .**



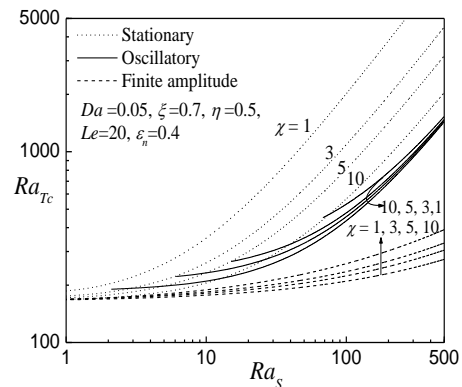
**Fig. 2. Neutral stability curves for different values of Damkohler number  $\chi$ .**

Figure 2 indicates the effect of the Damkohler number  $\chi$  on the neutral stability curves for stationary and oscillatory modes. We find that the minimum value of Rayleigh number for oscillatory mode increases with increasing Damkohler number  $\chi$ , indicating that the Damkohler number stabilizes the system in the oscillatory mode. On the other hand, increasing the Damkohler number decreases the minimum of the stationary Rayleigh number. This indicates that chemical reaction parameter destabilizes the system in the case of stationary mode. Thus, the chemical reaction parameter has a contrasting effect on stationary and oscillatory modes. It is also interesting to note that there exists critical Damkohler number  $\chi_c$  such that when  $\chi > \chi_c$ , the convection mode switches to the stationary type, and when  $\chi < \chi_c$ , the convection first sets in through oscillatory mode. Further, when  $\chi = \chi_c$ , both stationary and oscillatory convection occurs simultaneously but with different wavenumbers (e.g., for fixed values of

$Da = 0.05, \xi = 0.7, \eta = 0.5, Le = 20, Ra_S = 50$ , and  $\epsilon_n = 0.4$ , we find  $\chi_c = 8.59665$ , see Fig. 2). Further, we can observe from this figure that the effect of the Damkohler number  $\chi$  is more pronounced on the stationary mode than on the oscillatory mode.



**Fig. 3. Variation of critical Rayleigh number with solute Rayleigh number for different values of Darcy number  $Da$ .**

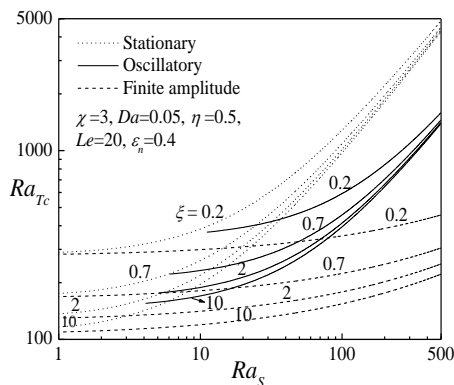


**Fig. 4. Variation of critical Rayleigh number with solute Rayleigh number for different values of Damkohler number  $\chi$ .**

The detailed behavior of critical Rayleigh number with respect to the solute Rayleigh number for stationary, oscillatory and finite amplitude modes is analyzed in the  $Ra_{Tc} - Ra_S$  plane through Figs. 3-8. We observe from these figures that all the quantities namely, the critical Rayleigh number for stationary, oscillatory, and finite amplitude modes are increasing functions of the solute Rayleigh number. It is clear that for the parameters chosen for these figures, the steady finite amplitude mode sets in prior to the oscillatory and stationary modes. Further, the steady finite amplitude Rayleigh number is a slowly increasing function of solute Rayleigh number. Furthermore, in each of these figures, the curves corresponding to the oscillatory convection start from a point where the solute Rayleigh number attains some threshold value below which the oscillatory convection is not possible. This threshold value depends on the other

parameters. Figure 3 shows the variation of  $Ra_{Tc}/Da$  with  $Ra_S$  for different values of the  $Da$ . We find that an increase in  $Da$  decreases the critical Rayleigh number  $Ra_{Tc}/Da$  for stationary, oscillatory and finite amplitude modes indicating that the effect is destabilizing. Also, we observe from this figure that the threshold value of the solute Rayleigh number below which the oscillatory convection is not possible increases with an increase of the Darcy number. The variation of the critical Rayleigh number with the solute Rayleigh number for different values of the Damkohler number  $\chi$  is shown in Fig. 4. The critical Rayleigh number for the oscillatory and finite amplitude modes increases with an increase in the value of  $\chi$ , indicating that the effect of Damkohler number is to delay the onset of convection in oscillatory and finite amplitude modes. On the other hand, the critical Rayleigh number for the stationary mode decreases with an increase of  $\chi$ , indicating that the effect of chemical reaction parameter is to destabilize the system in the stationary mode. Also, we find from this figure that the threshold value of the solute Rayleigh number below which the oscillatory convection is not possible increases with an increase of Damkohler number.

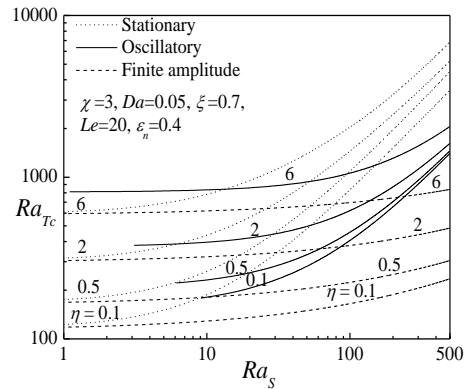
In Fig. 5 the variation of  $Ra_{Tc}$  with  $Ra_S$  for different values of mechanical anisotropy parameter  $\xi$  is shown. We find from this figure that the critical Rayleigh number decreases with an increase of mechanical anisotropy parameter  $\xi$  for all the cases namely stationary, oscillatory, and finite amplitude modes. Also, we observe that the threshold value of the solute Rayleigh number below which the oscillatory convection is not possible increases with the decrease of mechanical anisotropy parameter.



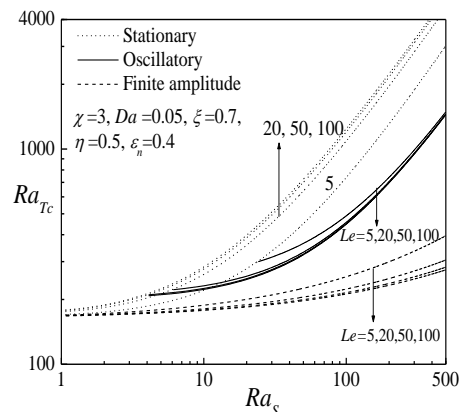
**Fig. 5. Variation of critical Rayleigh number with solute Rayleigh number for different values of mechanical anisotropy parameter  $\xi$ .**

Figure 6 depicts the effect of thermal anisotropy parameter  $\eta$  on the critical Rayleigh number for stationary, oscillatory and finite amplitude modes. We observe that an increase in  $\eta$  increases the critical Rayleigh number for stationary, oscillatory,

finite amplitude modes indicating that the effect of an increase of thermal anisotropy parameter is to inhibit the onset of stationary, oscillatory, finite amplitude convection. Here, it is interesting to note that the effect of  $\eta$  is opposite to that of  $\xi$ . Also, we find from this figure that the threshold value of the solute Rayleigh number below which the oscillatory convection is not possible increases with the decrease of thermal anisotropy parameter.



**Fig. 6. Variation of critical Rayleigh number with solute Rayleigh number for different values of thermal anisotropy parameter  $\eta$ .**

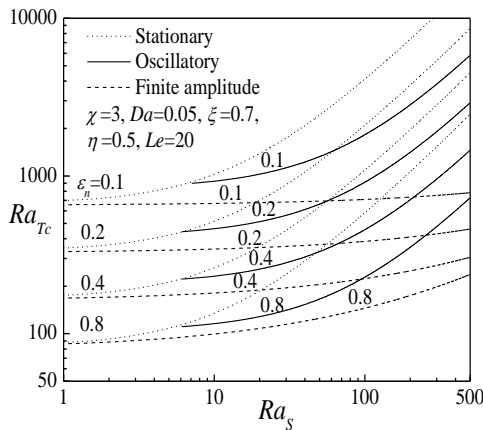


**Fig. 7. Variation of critical Rayleigh number with solute Rayleigh number for different values of Lewis number  $Le$ .**

The variation of  $Ra_{Tc}$  with  $Ra_S$  for different values of the Lewis number  $Le$  on the onset criteria is shown in Fig. 7. This figure reveals that with an increase of  $Le$  decreases the critical Rayleigh number for oscillatory and finite amplitude modes whereas it increases the critical Rayleigh number for stationary mode. Also, we observe that the threshold value of the solute Rayleigh number below which the oscillatory convection is not possible increases with the decrease of Lewis number. Figure 8 shows the effect of normalized porosity parameter  $\epsilon_n$  on  $Ra_{Tc}$  with  $Ra_S$  and for



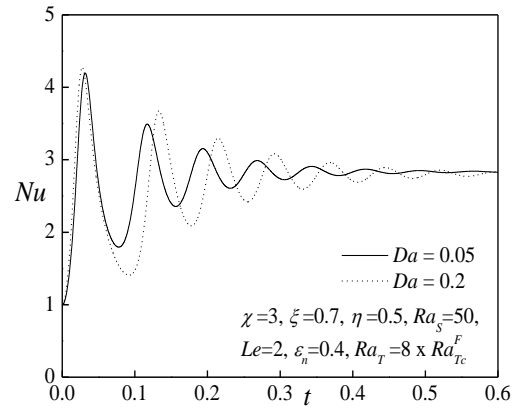
fixed values of the other parameters. We find that the critical Rayleigh number for the stationary, oscillatory, and finite amplitude modes decreases with an increase of  $\mathcal{E}_n$ , indicating that the effect of normalized porosity parameter  $\mathcal{E}_n$  is to advance the onset of stationary, oscillatory, and finite amplitude modes. As normalized porosity parameter increases, the thermal “lag” effect (double-advective behavior in the terminology of Phillips (2009)) is reduced. This makes advective heat transfer more effective and so makes it easier for the destabilizing thermal buoyancy gradient to produce convection.



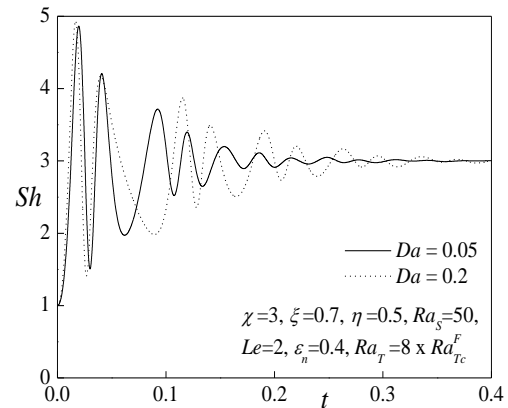
**Fig. 8.** Variation of critical Rayleigh number with solute Rayleigh number for different values of normalized porosity parameter  $\mathcal{E}_n$ .

To know the transient behavior of Nusselt and Sherwood numbers the autonomous system of unsteady finite-amplitude equations (35)-(39) is solved numerically using Runge-Kutta method with suitable initial conditions. The Nusselt number  $Nu$  and Sherwood number  $Sh$  are evaluated as a function of time  $t$ . The unsteady transient behavior of  $Nu$  and  $Sh$  is shown graphically through Figs. 9-13. From these figures it is seen that both  $Nu$  and  $Sh$  start with a conduction state value (i.e., 1) at  $t=0$  and then oscillate initially with time  $t$  and finally achieve a steady state value (i.e., close to 3) for  $t>0$ . This periodic variation of  $Nu$  and  $Sh$  is very short lived and decays as time progresses. The values of  $Nu$  and  $Sh$  then tend toward their steady state value 3. Figures 9(a) and 9(b) show that the transient heat and mass transfer increases with increasing both Darcy number  $Da$ . Figures 11(a) and 11(b) indicate that the effect of increasing mechanical anisotropy parameter is to decrease both heat and mass transfer. Figures 10(a), 10(b) and 13(a), 13(b) respectively show that Damkohler number  $\chi$  and normalized porosity parameter  $\mathcal{E}_n$  enhance the Nusselt number while suppress the Sherwood number. We find from Figures 12(a) and 12(b) show that increasing thermal anisotropy

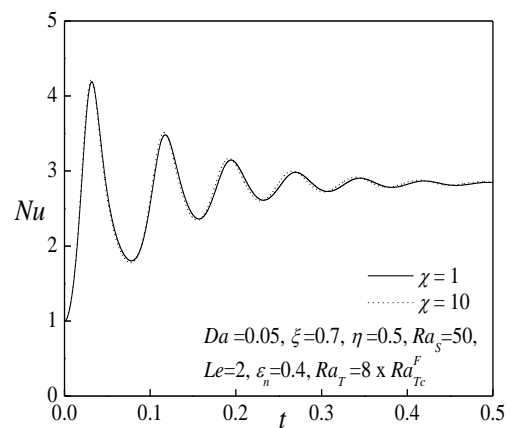
parameter  $\eta$  reduces the heat transfer while enhances the mass transfer.



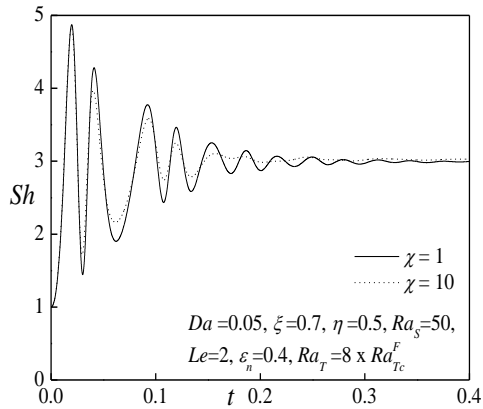
**Fig. 9(a).** Variation of Nusselt number with time for different values of  $Da$ .



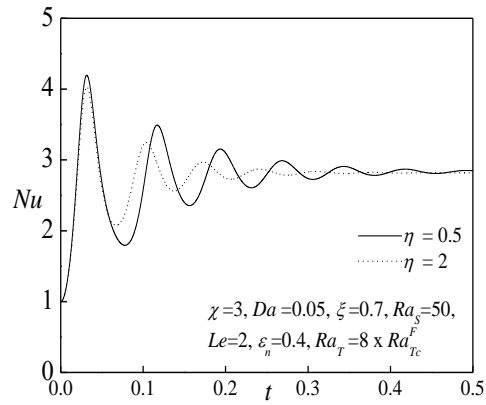
**Fig. 9(b).** Variation of Sherwood number with time for different values of  $Da$ .



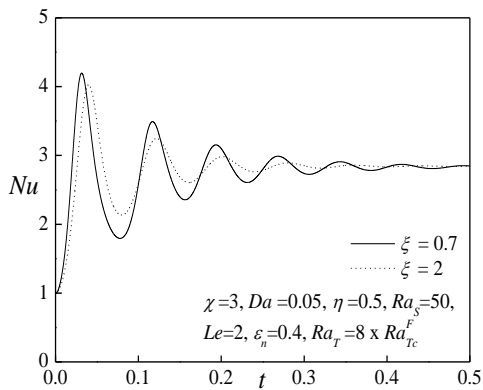
**Fig. 10(a).** Variation of Nusselt number with time for different values of  $\chi$



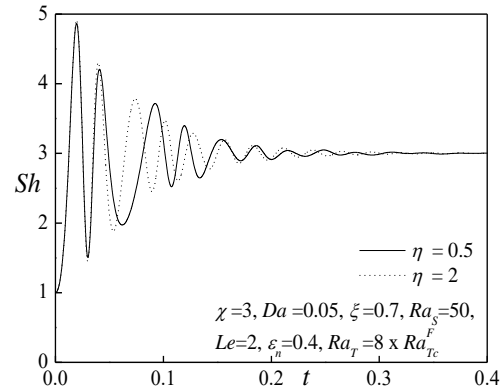
**Fig. 10(b).** Variation of Sherwood number with time for different values of  $\chi$ .



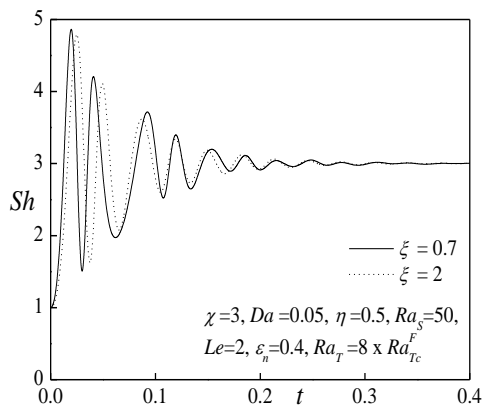
**Fig. 12(a).** Variation of Nusselt number with time for different values of  $\eta$ .



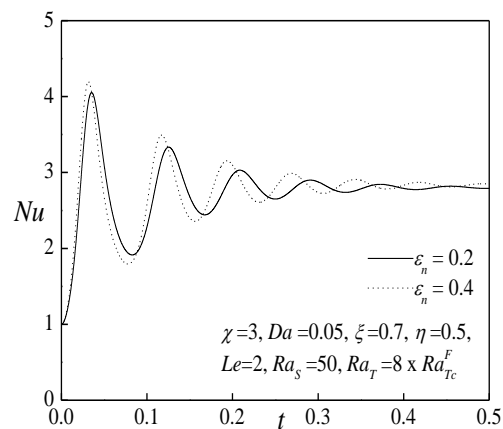
**Fig. 11(a).** Variation of Nusselt number with time for different values of  $\xi$ .



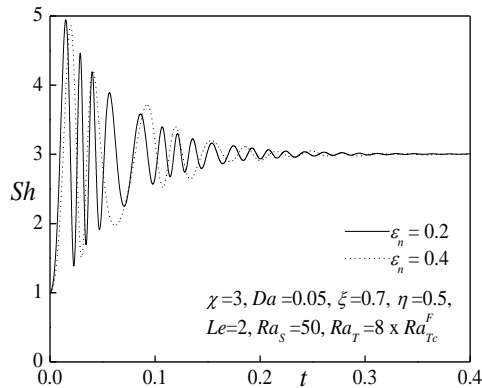
**Fig. 12(b).** Variation of Sherwood number with time for different values of  $\eta$ .



**Fig. 11(b).** Variation of Sherwood number with time for different values of  $\xi$ .



**Fig. 13(a).** Variation of Nusselt number with time for different values of  $\epsilon_n$ .



**Fig. 13(b). Variation of Sherwood number with time for different values of  $\varepsilon_n$ .**

## 6. CONCLUSIONS

The onset of double diffusive reaction-convection in a sparsely packed anisotropic porous layer is studied analytically using both linear and weak nonlinear stability analyses. The Darcy-Brinkman model is employed for the momentum equation. The linear theory which is based on the usual normal mode technique provides the onset criteria for both stationary and oscillatory convection. The nonlinear theory which is based on the truncated Fourier series technique provides a mean to measure the convection amplitudes and the rate of heat and mass transfer. The following conclusions are drawn.

1. The critical Rayleigh number for stationary, oscillatory, and finite amplitude modes are increasing functions of the solute Rayleigh number.
2. The Darcy number, mechanical anisotropy parameter and normalized porosity parameter have destabilizing effect, and the thermal anisotropy parameter has a stabilizing effect on the stationary, oscillatory, and finite amplitude convection.
3. The effect of increasing Damkohler number is to advance the onset of stationary mode and enhance the onset of oscillatory and finite amplitude modes.
4. The Lewis number has a stabilizing effect in the case of stationary mode while it has destabilizing effect in the case of oscillatory and finite amplitude modes.
5. The convective heat and mass transfer are suppressed by  $\xi$  where as enhanced by  $Da$ . The effects of increasing  $\chi$  and  $\varepsilon_n$  are to enhance the heat transfer while reduce the mass transfer. Heat transfer decreases while mass transfer increases with an increase of  $\eta$ . It is interesting to note that at large time  $t$ , both  $Nu$  and  $Sh$  approach the steady state value.

## ACKNOWLEDGEMENTS

One of the authors (M. Dhanraj) is grateful to UGC, New Delhi, for providing the financial assistance in the form of Research Fellowship in Science for Meritorious Students (RFSMS) under UGC-BSR Scheme No.4-1/2010(BSR) pt. IV, dated 06.01.2011.

## REFERENCES

- Bahloul, A., N. Boutana and P. Vasseur (2003). Double diffusive and Soret induced convection in a shallow horizontal porous layer. *J. Fluid Mech.* 491, 325-352.
- Bhadauria, B. S., I. Hashim, J. Kumar and A. Srivastava (2013). Cross diffusion convection in a Newtonian fluid-saturated rotating porous medium. *Transp. Porous Media* 98, 683-697.
- Gaikwad, S. N. and S. Kamble (2014). Linear stability analysis of double diffusive convection in a horizontal sparsely packed rotating anisotropic porous layer in presence of Soret effect. *Journal of Applied Fluid Mechanics* 7(3), 459-471.
- Gangadhar, K. (2013). Soret and Dufour effects on hydromagnetic heat and mass transfer over a vertical plate with a convective surface boundary condition and chemical reaction. *Journal of Applied Fluid Mechanics* 6(1), 95-105.
- Gatica, J. E., H. J. Viljoen and V. Hlavacek (1989). Interaction between chemical reaction and natural convection in porous media. *Chem. Eng. Sci.* 44(9), 1853-1870.
- Givler, R. C. and S. A. Altobelli (1994). A determination of the effective viscosity for the Brinkman Forchheimer flow model. *J. Fluid Mech.* 258, 355-370.
- Govender, S. (2006). On the effect of anisotropy on the stability of convection in a rotating porous media. *Transp. Porous Media* 64, 413-422.
- Ingham, D. B. and I. Pop (2005). *Transport Phenomena in Porous Media*, Elsevier, Oxford.
- Jupp, T. E. and A. W. Woods (2003). Thermally-driven reaction fronts in porous media. *J. Fluid Mech.* 484, 329-346.
- Kuznetsov, A. V. and D. A. Nield (2010). Thermal instability in a porous medium layer saturated by a nanofluid: Brinkman model. *Transp. Porous Media* 81, 409-422.
- Malashetty, M. S. (1993). Anisotropic thermo convective effects on the onset of double diffusive convection in a porous medium. *Int. J. Heat Mass Transf.* 36, 2397-2401.
- Malashetty, M. S. and B. S. Biradar (2011). The onset of double diffusive reaction-convection in an anisotropic porous layer. *Phys. Fluids* 23, 064102-1-12.

- Malashetty, M. S. and M. Swamy (2010). The onset of convection in a binary fluid saturated anisotropic porous medium. *Int. J. Therm. Sci.* 49, 867-878.
- Malashetty, M. S. and S. N. Gaikwad (2003). Onset of convective instabilities in a binary liquid mixtures with fast chemical reactions in a porous medium. *Heat Mass Transf.* 39, 415-420.
- Mamou, M. (2002). Stability analysis of double diffusive convection in porous enclosures, in D.B. Nield, D.A. (1968). Onset of thermohaline convection in a porous medium. *Water Resources Res.* 11, 553-560.
- Nield, D. A. (1968). Onset of thermohaline convection in a porous medium. *Water Resources Res.* 11, 553-560.
- Nield, D. A. and A. Bejan (2006). *Convection in Porous Media*. third ed. Springer, Berlin.
- Patil, P. R. and N. Rudraiah (1980). Linear convective stability and thermal diffusion of a horizontal quiescent layer of a two component fluid in a porous medium. *Int. J. Eng. Sci.* 18, 1055-1059.
- Phillips, O. M. (2009). *Geological Fluid Dynamics*. Cambridge University Press, New York.
- Poulikakos, D. (1986). Double diffusive convection in a horizontally sparsely packed porous layer. *Int. Commun. Heat Mass Transf.* 13, 587-598.
- Pritchard, D. and C. N. Richardson (2007). The effect of temperature-dependent solubility on the onset of thermosolutal convection in a horizontal porous layer. *J. Fluid Mech.* 571, 59-95.
- Rudraiah, N., P. K. Srimani and R. Friedrich (1982). Finite amplitude convection in a two component fluid saturated porous layer. *Int. J. Heat Mass Transf.* 25, 715-722.
- Shivakumara, I. S. and R. Sumithra (1999). Non-Darcian effects on double diffusive convection in a sparsely packed porous medium. *Acta Mechanica* 132, 113-127.
- Shivakumara, I. S., S. Sureshkumar and N. Devaraju (2012). Effect of non-uniform temperature gradients on the onset of convection in a couple stress fluid saturated porous medium. *Journal of Applied Fluid Mechanics* 5(1), 49-55.
- Steinberg, V. and H. Brand (1983). Convective instabilities of binary mixtures with fast chemical reaction in a porous medium. *J. Chem. Phys.* 78(5), 2655-2660.
- Steinberg, V. and H. Brand (1984). Amplitude equations for the onset of convection in a reactive mixture in a porous medium. *J. Chem. Phys.* 80(1), 431-435.
- Tyvand, P. A. (1980). Thermohaline instability in anisotropic porous media. *Water Resources Res.* 16, 325-330.
- Vadasz, P. (2008). *Emerging Topics in Heat and Mass Transfer in Porous Media*, Springer, Berlin.
- Vafai, K. (2000). *Handbook of Porous Media*, Marcel Dekker, New York.
- Vafai, K. (2005). *Handbook of Porous Media*, second ed. Taylor and Francis, London.
- Vanishree, R. K. (2014). Effects of through-flow and internal heat generation on a thermo convective instability in an anisotropic porous medium. *Journal of Applied Fluid Mechanics* 7(4), 581-590.
- Viljoen, H. J., J. E. Gatica and V. Hlavacek (1990). Bifurcation analysis of chemically driven convection. *Chem. Eng. Sci.* 45(2), 503-517.