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Fully Developed Flow of Fourth Grade Fluid through the Channel with Slip Condition in the Presence of a Magnetic Field

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ABSTRACT

In this paper, to study the incompressible fully developed flow of a non-Newtonian fourth grade fluid in a flat channel under an externally applied magnetic field, an appropriate analysis has been performed considering the slip condition on the walls. The governing equations, Ohm's law, continuity and momentum for this problem are reduced to a nonlinear ordinary form. The nonlinear equation with robin mixed boundary condition is solved with collocation (CM) and least square (LSM) methods**.** The effects of parameter**s** such as non-Newtonian, magnetic field and slip parameters on dimensionless velocity profiles will be discussed. In the end, the results could bring us to this conclusion that collocation and least square methods can be used for solving nonlinear differential equations with robin mixed condition.

Keywords: Collocation method; Channel; Fourth grade fluid; Least Square Method.

NOMENCLATURE

A	pressure gradient	u	velocity
A_n	Rivlin-Erickson tensors	$\tilde{u}$	approximate function of u
B	total magnetic field	V	velocity field
B_0	external magnetic field	W_i	weighted function
b	induced magnetic field		
c_i	constant of trial function	α_1, α_2	material constants
d	half distant of parallel plates	$\beta_1, \beta_2, \beta_3$	material constants
E	electric field	$\gamma_1 \cdots \gamma_8$	material constants
I	identity tensor	δ	delta function
Ha	Hartmann number	η	dimensionless parameter of channel width
N_f	non-Newtonian parameter		
P	pressure	λ	slip parameter
$p(x)$	a function	μ	viscosity
$R(x)$	Residual function	ρ	density
t	time	σ	electric conduction
		τ	stress tensor

1. INTRODUCTION

In recent years, the study on non-Newtonian fluids has gained importance and achieved great significance in the industry and technical operations (Siddiqui *et al.*, 2009; Choudhury and Kumar Das, 2014). The classical Navier–Stokes equations not able to describe and explain features of complex rheological fluids such as: food stuffs, shampoo, blood, synovial, paints, micro fluidics and polymer solutions. This kind of fluids is usually known non-Newtonian fluids, which unlike Newtonian fluids, the ratio of shear stress to shear rate is not linear. Many empirical and semi-empirical non-Newtonian models or constitutive equations have been proposed (Islam *et al.*, 2011). Among these, the fluids of differential type (Dunn and Rajagopal, 1995; Truesdell and Noll, 2004) have received considerable attention.

Because of their simplicity and originality, parallel plates are often used to simulate the actual flow domain conditions in some materials processing applications such as continuous casting, plastic forming and extrusion. Some research have been carried out to analyze the flow of different classes of materials in ducts and channels using various constitutive equations such as inelastic and linear/nonlinear viscoelastic models (Mokarizadeh *et al.*, 2013; Ali *et al.*, 2010; Hayat *et al.* 2006, Siddiqui *et al.*, 2006; Keimanesh *et al.*, 2011; Mohyuddin, 2005; Ramesh and Devakar, 2015).

Since get exact analytical solution for a nonlinear problem is not easy, we tend to semi-analytical solutions (Ganji and Languri, 2010; Hashemi Kachapi and Ganji, 2011; Nayfeh, 1985: Abbasi *et al.*,2014). There are some simple and accurate approximation techniques for solving differential equations called the Weighted Residuals Methods (WRMs). Collocation (CM), Galerkin (GM) and Least Square (LSM) are examples of the WRMs.

Actually, LSM and CM are two of the most effective and convenient solutions for both linear and nonlinear equations and don't require linearization or small perturbation. Motivated by these facts, we used LSM and CM to obtain the solutions of the fully developed steady flow of a fourth grade fluid between two stationary parallel plates with slip conditions at walls.

This paper by this powerful methods deal to solve nonlinear equation with Robin mixed condition. What is more, the results were evaluated and compared, and the numerical analysis validated and properly documented.

2. General Guidelines

Let us consider the fully developed laminar flow of an electrically conducting fourth grade fluid in a channel as shown in Fig.1. The slip boundary conditions are exerted on walls. The uniform magnetic field, B_0, is imposed along the y-axis. The governing equations, continuity, momentum and Ohm's law for the problem can be written as follows:

$$\nabla . V = 0, \tag{1}$$

$$\rho \frac{DV}{Dt} = -\nabla p + \operatorname{div} \tau + J \times B \tag{2}$$

$$J = \sigma(E + V \times B) \tag{3}$$

Where V is the velocity vector, ρ the constant density, ∇ the Nabla operator, p the pressure, τ the stress tensor, and D/Dt denotes the material derivative. The σ and J denote electrical conductivity and current density respectively and $B = B_0 + b$ (b being the induced magnetic field and B_0 an external magnetic field), is the total magnetic field and E is the electric field. It is assumed that the magnetic Reynolds Number is small and the induced magnetic field, b, due to the motion of the electrically conducting fluid is negligible. It is also assumed that the electrical conductivity of fluid, σ is constant and the external electric field is zero.

For the present model we take the velocity field of the form:

$$V = (u(y), 0, 0). \tag{4}$$

Under these assumptions the last term in Eq. (2), The Lorentz force per unit volume is given by:

$$J \times B = -\sigma B_0^2 u, \tag{5}$$

As discussed in (Siddiqui *et al.*, 2009; Islam *et al.* 2011), the stress tensor τ defining a fourth-grade fluid is given by

$$\tau = \sum_{i=1}^{4} S_i, \tag{6}$$

where

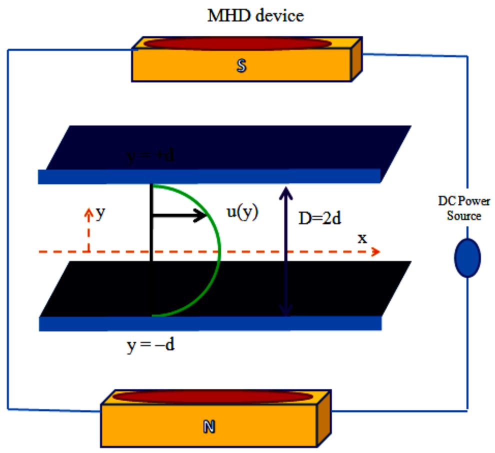


Fig. 1. Schematic diagram of the physical system.

$$S_0 = -pI, \qquad S_1 = \mu A_1, \qquad S_2 = \alpha_1 A_2 + \alpha_2 A_1^2,$$
$$S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (tr(A_1)) A_1,$$
$$S_4 = \gamma_1 A_4 + \gamma_2 (A_3 A_1 + A_1 A_3) + \gamma_3 A_2^2$$
$$+ \gamma_4 (A_2 A_1^2 + A_1^2 A_2) + \gamma_5 (tr A_2) A_2 + \gamma_6 (tr A_2) A_1^2$$
$$+ (\gamma_7 (tr A_3) + \gamma_8 (A_2 A_1)) A_1, \tag{7}$$

where I is the identity tensor, μ, α_1, α_2, β_1, β_2, β_3, $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$, and γ_8 are material constants. The Rivlin-Ericksen tensors A_n (Siddiqui *et al.*, 2009; Mohyuddin, 2005) are defined by

$$A_0 = I$$
$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\nabla V) + (\nabla V)^t A_{n-1}, \qquad n \geq 1, \tag{8}$$

which t is the transpose symbol.

The continuity (Eq. (1)) is satisfied by Eq. (6) and Eq. (2) can be written in component form as:

$$\mu\left(\frac{d^2}{dy^2}u(y)\right)+6(\beta_2+\beta_3)\left(\frac{d}{dy}u(y)\right)^2\left(\frac{d^2}{dy^2}u(y)\right)$$
$$-\sigma\mu^2 u(y)B_0^{\,2}=\left(\frac{\partial}{\partial x}p(x,y)\right) \qquad (9)$$

Y-component:

$$2\alpha_2\left(\frac{d}{dy}u(y)\right)\left(\frac{d^2}{dy^2}u(y)\right)+8\gamma_6\left(\frac{d}{dy}u(y)\right)^3\left(\frac{d^2}{dy^2}u(y)\right)$$
$$=\left(\frac{\partial}{\partial y}p(x,y)\right) \qquad (10)$$

Because the flow is fully developed, the left side of Eq. (9) is only a function of y. If we differentiating both side of Eq. (10) with respect to x and then integrating with respect to y, we can see that the right side of Eq. (9) is only function of x. Thus, Eq. (9) is valid if it's both side equal to a constant. So,

$$\mu\left(\frac{d^2}{dy^2}u(y)\right)+6(\beta_2+\beta_3)\left(\frac{d}{dy}u(y)\right)^2\left(\frac{d^2}{dy^2}u(y)\right)$$
$$-\sigma\mu^2 u(y)B_0^{\,2}=\frac{dp^*}{dx}=A \qquad (11)$$

where $\beta=\beta_2+\beta_3$ and A is a constant. Therefore, the problem reduces to solve the second-order nonlinear ordinary differential. Due to symmetry and slip conditions at either of the two plates, there are the following boundary conditions:

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=d}=-\lambda\, u_{(d)} \qquad (12)$$

By introducing the following non-dimensional parameters:

$$\eta=\frac{y}{d},\quad U(\eta)=\frac{\mu u(y)}{Ad^2},\quad N_f=\frac{A^2d^2\beta}{\mu^3},\quad Ha=B_0 d\sqrt{\frac{\sigma}{\mu}} \qquad (13)$$

Substituting these functions into Eq. (11) and Eq. (12), rewriting these equations, we finally obtain the following system of nonlinear equations:

$$\frac{d^2U}{d\eta^2}+6N_f\left(\frac{dU}{d\eta}\right)^2\frac{d^2U}{d\eta^2}-Ha^2\,U-1=0, \qquad (14)$$

$$\left.\frac{dU}{d\eta}\right|_{\eta=0}=0, \qquad \left.\frac{dU}{d\eta}\right|_{\eta=1}=-\lambda.U(1) \qquad (15)$$

2.1 Collocation Method (CM)

Suppose we have a differential operator D is acted on a function u to produce a function p (Hatami *et al.* 2013):

$$D(u(x))=p(x) \qquad (16)$$

We wish to approximate u by a function $\tilde{u}$, which is a linear combination of basic functions chosen from a linearly independent set. That is:

$$u\cong\tilde{u}=\sum_{i=1}^{n}C_i\varphi_i \qquad (17)$$

Now, by substituting Eq. (17) into the differential operator D, the result of the operations is not $p(x)$ in general. Hence an error or residual will exist:

$$E(x)=R(x)=D(\tilde{u}(x))-p(x)\neq 0 \qquad (18)$$

The notion in the collocation is to force the residual to zero in some average sense over the domain. That is:

$$\int_x R(x)\,W_i(x)=0 \qquad i=1,2,...,n \qquad (19)$$

Where the number of weight functions W_i are identically equal the number of unknown constants C_i in $\tilde{u}$. The result is a set of n algebraic equations for the unknown constants C_i. For collocation method, the weighting functions are taken from the family of Dirac δ functions in the domain. That is, $W_i(x)=\delta(x-x_i)$. The Dirac δ function has the property that:

$$\delta(x-x_i)=\begin{cases}1 & if\ x=x_i\\ 0 & Otherwise\end{cases} \qquad (20)$$

And residual function must force to be zero at specific points.

2.2 Application of CM on Problem

Consider the trial function as:

$$u(\eta)=\frac{1}{2}c_0\lambda\eta^2+\frac{1}{3}c_1\lambda\eta^3+\frac{1}{4}c_2\lambda\eta^4$$
$$+\frac{1}{5}c_3\lambda\eta^5+c_0\left(-\frac{1}{2}\lambda-1\right)+c_1\left(-\frac{1}{3}\lambda-1\right) \qquad (21)$$
$$+c_2\left(-\frac{1}{4}\lambda-1\right)+c_3\left(-\frac{1}{5}\lambda-1\right)$$

The trial function satisfies the boundary condition in Eq. (15) and setting into Eq. (14), residual function,

$R(c_0,c_1,c_2,c_3,\eta)$, is found as:

$$R(\eta)=1+24N_f\,c_0^2\lambda^3\eta^3c_1+30N_f\,c_0^2\lambda^3\eta^4c_2+Ha^2c_2$$
$$+30N_f\,c_0\lambda^3\eta^4c_1^2+54N_f\,c_0\lambda^3\eta^8c_3^2\,42N_f\,c_1^2\lambda^3\eta^6c_2$$
$$+48N_f\,c_1^2\lambda^3\eta^7c_3+48N_f\,c_1\lambda^3\eta^7c_2^2+60N_f\,c_1\lambda^3\eta^9c_3^2$$
$$+60N_f\,c_2^2\lambda^3\eta^9c_3\,66N_f\,c_2\lambda^3\eta^{10}c_3^2+12N_f\,c_1^3\lambda^3\eta^5$$
$$+6N_f\,c_0^3\lambda^3\eta^2+24N_f\,c_3^3\lambda^3\eta^{11}-0.333Ha^2c_1\lambda\eta^3$$
$$+84N_f\,c_0\,\lambda^3\eta^6c_1c_3-0.2Ha^2c_3\lambda\eta^5+Ha^2c_0+Ha^2c_1$$
$$+Ha^2c_3+96N_f\,c_0\lambda^3\eta^7c_2c_3-0.25Ha^2c_2\lambda\eta^4$$
$$72Nfc_0\lambda^3\eta^5c_1c_2+c_0\lambda+0.5Ha^2c_0\lambda+0.33Ha^2c_1\lambda$$
$$+2c_1\lambda\eta+0.2Ha^2c_3\lambda+3c_2\lambda\eta^2-0.5Ha^2c_0\lambda\eta^2$$
$$+36N_f\,c_0^2\lambda^3\eta^5c_3+0.25Ha^2c_2\lambda+18N_f\,c_2^3\lambda^3\eta^8$$
$$+108N_f\,c_1\,\lambda^3\eta^8c_2c_3+42N_f\,c_0\lambda^3\eta^6c_2^2+4c_3\lambda\eta^3=0 \qquad (22)$$

Table 1 Determined values of unknown constants C_i at various N_f and Ha for λ

Method	Constant	$Nf=0.1$		$Nf=1$	
		$Ha=0.1$	$\lambda=0.1$	$Ha=2$	$\lambda=0.9$
CM	C_0	9.15886202100		0.0976611706200	
	C_1	0.02124813622		0.0002592917211	
	C_2	1.66559390900		0.0592753897500	
	C_3	0.631027785400		0.0114375519000	
LSM	C_0	9.1553332060		0.098136953020	
	C_1	0.04151502341		0.003092236272	
	C_2	1.59265116100		0.066764011040	
	C_3	0.57810309220		0.006746087832	

On the other hands, the residual function must be close to zero. For reaching this importance, for specific points in the domain $\eta\in[0,1]$ should be chosen. These points are selected as:

$$R\left(\frac{1}{5}\right)=0,\quad R\left(\frac{2}{5}\right)=0,\quad R\left(\frac{3}{5}\right)=0,\quad R\left(\frac{4}{5}\right)=0 \tag{23}$$

For example the first equation is written as:

$$\begin{aligned}a_1 &= -1+\frac{12}{25}Ha^2c_0\lambda+\frac{18}{390625}N_f c_2^3\lambda^3+\frac{3124}{15625}Ha^2c_3\lambda\\
&+\frac{12}{3125}N_f c_1^3\lambda^3+\frac{124}{375}Ha^2c_1\lambda+\frac{24}{125}N_f c_0^2\lambda^3c_1+Ha^2c_2\\
&+\frac{6}{125}N_f c_0^2\lambda^3c_2+\frac{36}{3125}N_f c_0^2\lambda^3c_3+\frac{6}{125}N_f c_0\lambda^3c_1^2\\
&+\frac{42}{15625}N_f c_0\lambda^3c_2^2+\frac{54}{390625}N_f c_0\lambda^3c_3^2+\frac{42}{15625}N_f c_1^2\lambda^3c_2\\
&+\frac{48}{78125}N_f c_1^2\lambda^3c_3+\frac{48}{78125}N_f c_1\lambda^3c_2^2+\frac{156}{625}Ha^2c_2\lambda\\
&+\frac{12}{390625}N_f c_1\lambda^3c_3^2+\frac{12}{390625}N_f c_2^2\lambda^3c_3+Ha^2c_1+\frac{6}{25}N_f c_0^3\lambda^3\\
&+\frac{72}{3125}N_f c_0\lambda^3c_1c_2+\frac{96}{78125}N_f c_0\lambda^3c_2c_3+\frac{84}{15625}N_f c_0\lambda^3c_1c_3\\
&+\frac{108}{390625}N_f c_1\lambda^3c_2c_3+\frac{4}{125}c_3\lambda+\frac{2}{5}c_1\lambda+\frac{3}{25}c_2\lambda+Ha^2c_3\\
&+\frac{66}{9765625}N_f c_2\lambda^3c_3^2+\frac{24}{48828125}N_f c_3^3\lambda^3+Ha^2c_0+c_0\lambda=0\end{aligned} \tag{24}$$

The rest of equations are written similarity. Finally by substitutions the λ, N_f and Ha into the residual function, $R(c_0,c_1,c_2,c_3,\eta)$, a set of four equations and four unknown coefficients are obtained. After solving these equations for unknown parameters (c_0,c_1,c_2,c_3), the velocity distribution equation will be determined that shows in table 1.

3.1. Principles of Least Square Method

If the continuous summation of all the squared residuals is minimized, the rationale behind the name can be seen. In other words:

$$S=\int_X R(x)R(x)\,dx=\int_X R^2(x)\,dx \tag{25}$$

In order to achieve a minimum of this scalar function, the derivatives of *S* with respect to all the unknown parameters must be zero. That is,

$$\frac{\partial S}{\partial c_i}=2\int_X R(x)\frac{\partial R}{\partial c_i}dx=0 \tag{26}$$

Comparing with Eq. (19), the weight functions are seen to be

$$W_i=\frac{\partial R}{\partial c_i} \tag{27}$$

However, the ‘‘2’’ coefficient dropped, since it cancels out in the equation. Therefore the weight functions for the least squares method are just the derivatives of the residual with respect to the unknown constants

3.2. Application

Consider the trial function and corresponding residual as Eq. (21) and Eq. (22) and using Eq. (27), the first weight function is obtained as,

$$\begin{aligned}\frac{\partial R}{\partial c_0}&=6.561\eta^4c_1^2+10.4976c_0\eta^3c_1+11.8098\eta^8c_3^2\\
&+13.122c_0\eta^4c_2+4.1625-1.0125\eta^2+9.1854\eta^6c_2^2\\
&+15.7464c_0\eta^5c_3+3.9366c_0^2\eta^2+15.7464\eta^5c_1c_2\\
&+18.3708\eta^6c_1c_3+20.9952\eta^7c_2c_3\end{aligned} \tag{28}$$

3. Result

In this study, we employed CM and LSM to find the velocity field for fully developed steady flow of a fourth grade fluid between two stationary parallel plates under transverse magnetic filed. The solutions are shown graphically, because they were too long to be mentioned here.

The comparison of results between the applied methods, CM and LSM and Numerical Methods, for different values of active parameters is shown in Figure. 2. The numerical solution is performed using the algebra package Maple 16.0, to solve the present case. The package uses a fourth order Runge–Kutta procedure for solving nonlinear boundary value (B-V) problem (Hatami *et al.* 2013; Hatami and Ganji 2014). Validity of LSM is shown in Table 2. In these tables, the Error is defined as:

$$Error = \left| U(\eta)_{NUM} - U(\eta)_{Analytical} \right| \tag{29}$$

The results show that the solution methods are precise and this investigation is completed by depicting the effects of some important parameters by Least Square Method to evaluate how these parameters influence on this fluid.

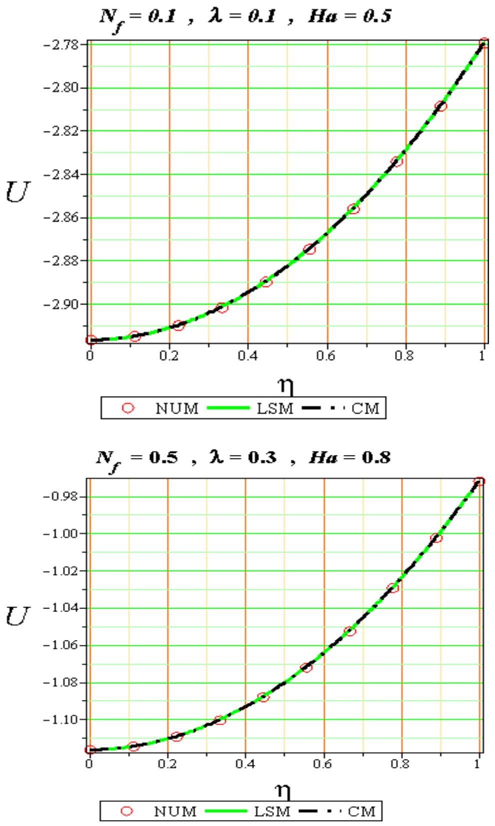


Fig. 2. The comparison between the Numerical, CM and LSM solution for U .

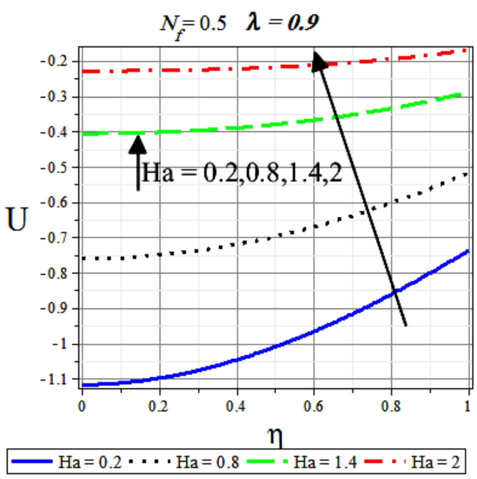


Fig. 3. Dimensionless velocities predicted by LSM in different Ha number when $N_f = 0.5, \lambda = 0.1$

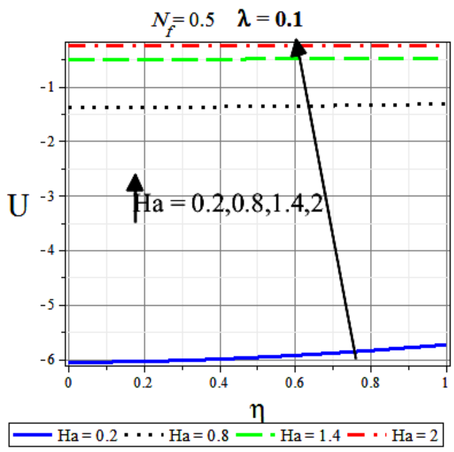
As shown, an increase in the magnetic parameter leads to decrease in the velocity components at given point as can be seen from Fig. 3. This is due to the fact that applied transverse magnetic field produces a drag in the form of Lorentz force thereby decreasing the magnitude of velocity. In addition, Fig. 4 shows the effect of non-Newtonian parameter N_f on the velocity components for $\lambda = 0.5, Ha = 1$. It is noticed that an increase in dimensionless parameters N_f tends to decrease the velocity profile $U(\eta)$. It is worth mention that, the same effect is observed for the slip parameter which is depicted by the Fig. 5. This is due to the fact that, with the increasing of slip parameter some part of fluid molecules strike solid surface and reflected diffusely increases then velocity decreases

Table 2 The results of CM, LSM and Numerical methods for $U(\eta)$ for $\lambda = 0.4$, $Ha = 1$ and $N_f = 1$

η	LSM	NUM	Error of LSM
0.00	-0.756148161	-0.756099597	4.85644E-05
0.10	-0.754928597	-0.754893640	3.49565E-05
0.20	-0.751266500	-0.751259302	7.19730E-06
0.30	-0.745151678	-0.745171879	2.02009E-05
0.35	-0.741169625	-0.741200541	3.09159E-05
0.40	-0.736567863	-0.736606669	3.88052E-05
0.50	-0.725493618	-0.725538967	4.53494E-05
0.55	-0.719014744	-0.719058962	4.42180E-05
0.60	-0.711903466	-0.711944071	4.06051E-05
0.65	-0.704156173	-0.704191206	3.50333E-05
0.70	-0.695769164	-0.695797279	2.81145E-05
0.75	-0.686738689	-0.686759201	2.05125E-05
0.80	-0.677060964	-0.677073886	1.29221E-05
0.85	-0.666732216	-0.666738245	6.02900E-06
0.90	-0.655748735	-0.655749190	4.54900E-07
0.95	-0.644106858	-0.644103634	3.22370E-06
1.00	-0.631802927	-0.631798488	4.43805E-06

4. CONCLUSION

In this study, a fully developed steady flow of a fourth grade fluid between two stationary parallel plates was analyzed using Least Square Method (LSM) and Collocation Method (CM). LSM and CM does not require small parameters in the equation so that the limitations of the traditional perturbation methods can be eliminated and thereby the calculations are straightforward. Effects of different physical parameters such as Slip number, the Non-Newtonian number and the magnetic field parameter on the velocity profiles of the problem have been investigated. As an important outcome from the present study, it can be observed that the results of LSM are more accurate than CM and they are in excellent agreement with numerical ones, so LSM can be used for finding analytical solutions of non-Newtonian problems easily. Also it can be concluded that increasing the magnetic parameter leads to decrease in velocity values in whole domain. In addition, increasing in slip parameter caused a decrease in velocity components to. It was shown that proposed methods provide simple, accurate and appropriate techniques for solving nonlinear differential equations with Robin mixed.



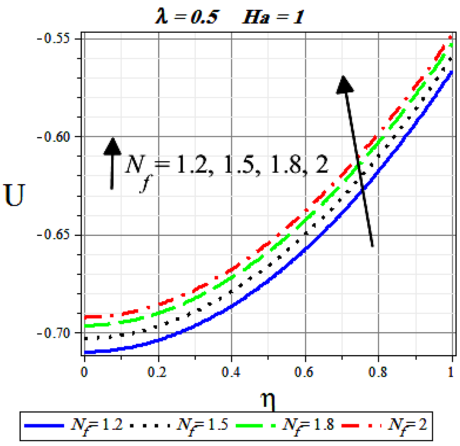


Fig. 4. Dimensionless velocities predicted by LSM in different N_f number when $\lambda = 0.5, Ha = 1$.

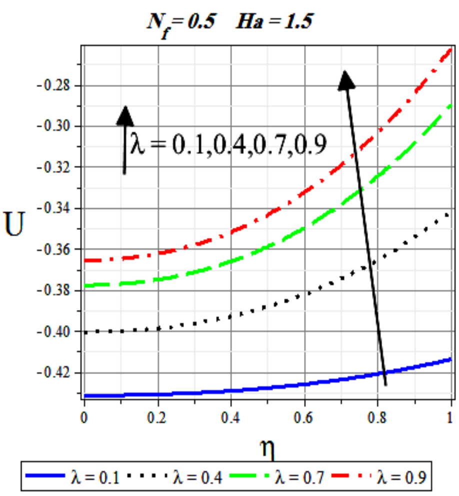


Fig. 5. Dimensionless velocities predicted by LSM in different λ number when $N_f = 0.5, Ha = 1.5$.

REFERENCES

Abbasi, M., D. D. Ganji and M. R. Taeibi (2014). MHD flow in a channel using new combination of order of magnitude technique and HPM. *Technical Gazette 21(2), 317-321*.

Ascher, U., R. Mattheij and R. Russell (1995). Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. 13 of classics in applied mathematics. SIAM

Aziz, A. (2006). *Heat Conduction with Maple*, R.T. Edwards, Philadelphia (PA).

Choudhuty, R. and S. Kumar Das (2014). Visco-Elastic MHD Free Convective Flow through Porous Media in Presence of Radiation and Chemical Reaction with Heat and Mass Transfer. *Journal of Applied Fluid Mechanics 7(4), 603-609.*

Dunn, J. E. and K. R. Rajagopal (1995). Fluids of differential type: Critical review and thermodynamic analysis. *International Journal of Engineering Science 33(5), 689–729*.

Ganji, D. D. and E. Mohseni Languri (2010). *Mathematical methods in nonlinear heat transfer* (A semi-analytical approach), Xlibris Corporation, USA.

Hashemi Kachapi, S. H. and D. D. Ganji (2011). *Analysis of nonlinear equations in fluids*. Vol. 3 of progress in nonlinear science. Asian academic publisher limited, Room 3208, Central Plaza, 18 Harbour Road, Wanchai, Hongkong, China, 1-294.

Hatami, M., R. Nouri and D. D. Ganji (2013). Forced convection analysis for MHD Al2O3–water nano fluid flow over a horizontal plate. *Journal of Molecular Liquids 187, 294–301*.

Hayat, T., N. Ahmed and M. Sajid (2006). Analytic solution for MHD flow of a third order fluid in a porous channel. *Journal of Computational and Applied Mathematics, 214 (2008), 572–582.*

Islam, S., Z. Bano, I. Siddique and A. M. Siddiqui (2011). The optimal solution for the flow of a fourth-grade fluid with partial slip. *Computers and Mathematics with Applications 61, 1507–1516.*

Keimanesh, M., M. M. Rashidi, A. J. Chamkha, R. Jafari (2011). Study of a third grade non-Newtonian fluid flow between two parallel plates using the multi-step differential transform method. *Computers and Mathematics with Applications 62, 2871–2891.*

Mohyuddin, M. R. (2005). *On solutions of non-linear equations arising in Rivlin-Ericksen fluids,* PhD Thesis, Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

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Table 3 The results of CM, LSM and Numerical methods for $U(\eta)$ for $\lambda = 0.4, Ha = 1$ and $N_f = 1$

η	CM	NUM	Error of CM
0.00	-0.755435424	-0.756099597	0.000664173
0.10	-0.754233549	-0.754893640	0.000660091
0.20	-0.750610552	-0.751259302	0.000648750
0.25	-0.747882989	-0.748523770	0.000640781
0.30	-0.744540374	-0.745171879	0.000631505
0.40	-0.735996956	-0.736606669	0.000609712
0.50	-0.724954241	-0.725538967	0.000584726
0.55	-0.718487503	-0.719058962	0.000571460
0.60	-0.711386168	-0.711944071	0.000557903
0.65	-0.703646980	-0.704191206	0.000544226
0.70	-0.695266681	-0.695797279	0.000530598
0.75	-0.686242013	-0.686759201	0.000517188
0.80	-0.676569720	-0.677073886	0.000504166
0.85	-0.666246544	-0.666738245	0.000491701
0.90	-0.655269227	-0.655749190	0.000479963
0.95	-0.643634512	-0.644103634	0.000469122
1.00	-0.631339143	-0.631798488	0.000459346

(2013). Heat transfer in Couette-Poiseuille flow between parallel plates of the Giesekus viscoelastic fluid. *Journal of Non- Newtonian Fluid Mechanics 196, 95–101*.

Nayfeh, A. H. (1985*). Problems in perturbation*, John Wiley and Sons, New York, USA.

Ramesh, K. and M. Devakar (2015). The Influence of Heat Transfer on Peristaltic Transport of MHD Second Grade Fluid through Porous Medium in a Vertical Asymmetric Channel. *Journal of Applied Fluid Mechanics 8(3), 351-365.*

Siddiqui, A. M., A. Zeb, Q. K. Ghori and A. M. Benharbit (2006). Homotopy perturbation method for heat transfer flow of a third grade fluid between parallel plates, Chaos. *Solitons and Fractals 36, 182–192.*

Siddiqui, A. M., M. Hameed, B. M. Siddiqui and Q. K. Ghori (2009). Use of Adomian decomposition method in the study of parallel plate flow of a third grade fluid. *Commun Nonlinear Sci Numer Simulat 15, 2388–2399.*

Truesdell, C. and W. Noll (2004). *The non-linear field theories of mechanics*, 3rd ed. Springer, USA.