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Flow of a Second Grade Fluid through Constricted Tube using Integral Method

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ABSTRACT

The steady flow of a second grade fluid through constricted tube for mild stenosis is modeled and analyzed theoretically. The governing equations are simplified by implying an order-of-magnitude analysis. Based on Karman Pohlhausen procedure polynomial solution for velocity profile is presented. Expressions for pressure gradient, shear stress, separation and reattachment points are also calculated. The effects of nondimensional parameters emerging in the model on the velocity profile, shear stress, pressure gradient are discussed and depicted graphically. The effect of non-Newtonian parameter on velocity profile, wall shear stress and pressure gradient is also analyzed. It is found that the Reynolds number strongly effect the wall shear stress, separation and reattachment points.

Keywords: Non-Newtonian fluids; Constricted tube; Analytical solutions; Shear stress; Back flow; Separation and reattachment points.

NOMENCLATURE

A_1, A_2	Rivlin-Erickson tensors	α_1, α_2	material constants
h, h	generalized pressure	μ	viscosity
$\tilde{r}, r, \tilde{z}, z$	axis	ρ	density
R_e	Reynolds number	$\tau_{\tilde{w}}, \tau_w$	shear stress
u, u, w, w	velocity components	∇	nebla
U	centreline velocity	∇^2	Laplacian parameter
Q	flux		

1. INTRODUCTION

A stenosis, localized narrowing in an arterial system of mammals, disturb normal blood flow through the artery and causes arterial disease. Hydrodynamic factors play a significant role in the development and progression of this disease. Flow characteristics of blood such as pressure, wall shearing stress, vortices and turbulence may have potential medical significance. In the vicinity of a stenosis, a knowledge of the flow characteristics may help to supplement the understanding of some of the major complications which can arise from such constrictions is quoted by Roach (1963) and Rodbard (1966). These include, an intravascular clot (the development of a thrombus), an ingrowth of tissue into the artery and the weakening and bulging of the artery downstream from the stenosis (post-stenotic dilatation). Once a vascular lesion has developed, there may be a coupling effect between its further development and changed flow characteristic Forrester (1968). Some constrictions are approximately axisymmetric or collar-like which may be caused by intravascular plaques or the impingement of ligament or spurs on the vessel wall (Roach (1963); Rodbard (1966)). Many researchers have pointed out that atherosclerotic plaques are caused primarily by faulty lipid metabolism. As atherosclerotic lesions are commonly found in curved arterial segments, at the entrance of branching vessels, or generally at locations of abrupt changes in geometry, which should take into account the flow characteristics of the blood. Due to separation of the main flow from the walls of arteries, static zones occur in the arterial system (Fox and Hugh (1966)). They proposed that there is an interaction of platelets and fibrin in these separation zones to form a mesh in which lipid particles become trapped with the subsequent formation of a plaque of atheroma. One of the major complications of prosthetic heart valve is the formation of thrombus. Numerous researchers have anticipated that stagnation zones near the valve contribute to the formation of thrombi. It, therefore, seems reasonable to venture that if separation regions of relatively stagnant flow occur near constriction, they may well contribute to the problems of thrombosis. The study of the mechanics of flow in constricted tubes remains a challenging problem though the potential importance of hydrodynamic has been documented for many years. Young (1968) has studied in detail the flow in a stenosis taking the flow in a mildly constricted tube based on a highly simplified linear model. Forrester and Young (1970) extended this work to embrace the effects of flow separation on a mild constriction. The problem of flow in constricted tubes was analyzed numerically by Lee and Fung (1970). Many physicians, researchers and scientists have made their efforts to understand the mechanics of fluid flow in constricted arteries considering the blood as Newtonian fluid. The blood, however, behaves like a Newtonian fluid only under certain conditions, of course, at low shear stress it becomes as a non-Newtonian fluid (Lee and Fung (1970)). The blood, however, can be treated as an incompressible Newtonian fluid at the flow rates encountered in the larger arteries where constriction commonly occur. Fox and Hugh (1966) proposed that the atherosclerosis plaque (constriction) is caused by intravascular clotting. For the first time Fry (1966) reported the endothelial changes by inserting a plug in the thoratic aorta of mongrel dogs, which abrupt the blood flow. Further, he obtained theoretical results for unsteady, axisymmetric, incompressible Newtonian fluid flow numerically and compared with the experimental one. Young (1968) reported the time dependent constriction in tube for viscous flow. Forrester and Young (1970) developed theoretical and experimental results for the blood flow through constricted tube. A primary goal of their research was to predict analytically the separation of flow at Reynolds number in constricted tube. For analytical results they use an integral method. An experiment was performed to check the theory which was valid for water and glycerol-water solution (viscous fluid) but not valid for the whole blood, as blood behaves as a non-Newtonian fluid at low shear rate. Further results were obtained for pressure drop, separation and reattachment regions and compared with the theoretical one. Morgan and Young (1974) investigated the development of a relatively simple and approximate solution of the fluid flowing through an axisymmetric artery having cosine shape constriction, which is valid for both, mild and sever constriction. Their general approach was an extension and modification of the work done by Forrester and Young (1970) and made use of both integral-momentum and integral-energy equation for viscous, steady and incompressible fluid. Haldar (1991) has analyzed the blood flow treating it as a Newtonian fluid flowing through an axisymmetric artery having constriction. Chow and Soda (1972) and Chow *et al*. (1971) explored various information related to the fluid flowing through an axisymmetric artery having cosine shape constriction. Nadeem and Akbar (2013) has anlayzed the flow of second grade fluid through a tapered artery using simulations. Mirza *et al.* (2013) analyzed steady and incompressible Newtonian fluid flowing through axisymmetric constricted tube taking constant volume flow rate. They discussed the stream lines, wall shear stress, pressure, separation and reattachment points, velocity profile and temperature distribution. In the present study we have the work (Forrester and Young (1970)) by taking the blood as non-Newtonian fluid of constant density flowing through an axisymmetric artery with a cosine shape constriction of constant volume flow rate. A mathematical model for a mild constriction (i. e δ / z_0 and δ / R_0 are very small) is developed and analyzed using integral approach. An order-of-magnitude analysis is employed to simplify the governing equation of motion of the problem under consideration. The flow is assumed to be steady just to make the problem mathematically manageable. However, it is anticipated that the steady flow result will be responsible for useful information. As arterial flow is pulsatile, this assumption, of course, can not be justified totally. Moreover, it is believed that blood flow, except in the ascending aorta or under pathological circumstances, is laminar. Also, it is well established that for a better understanding of the relationship between the development of constriction and blood flow in arteries, a knowledge of the flow characteristics in constriction is a necessary prerequisite. In this work, therefore, velocity, pressure, shear stress, separation and reattachment points of fluid flowing through constricted tube are analyzed.

2. GOVERNING EQUATIONS

The basic equations that govern the flow of an incompressible fluid consist of the conservation of mass and momentum and in the absence of body forces are given as

$$\tilde{\nabla}.\widetilde{\boldsymbol{V}} = 0 \tag{1}$$

$$\rho\left(\frac{\partial\widetilde{\boldsymbol{V}}}{\partial t} + \frac{\tilde{\nabla}\widetilde{\boldsymbol{V}}^2}{2} - \tilde{\nabla}\times\left(\tilde{\nabla}\times\widetilde{\boldsymbol{V}}\right)\right) = -\tilde{\nabla}\tilde{p} + \mu\tilde{\nabla}.\widetilde{\boldsymbol{A}}_{\boldsymbol{1}} + \alpha_1\tilde{\nabla}.\left(\frac{\partial\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}}{\partial t}\right) + \alpha_1\left(\left(\widetilde{\boldsymbol{V}}.\tilde{\nabla}\right)\tilde{\nabla}.\widetilde{\boldsymbol{A}}_{\boldsymbol{1}} + \left(\tilde{\nabla}\widetilde{\boldsymbol{V}}\right)^T\tilde{\nabla}.\widetilde{\boldsymbol{A}}_{\boldsymbol{1}} + \widetilde{\boldsymbol{A}}_{\boldsymbol{1}}.\tilde{\nabla}\left(\tilde{\nabla}\widetilde{\boldsymbol{V}}\right)^T\right) + (\alpha_1+\alpha_2)\tilde{\nabla}.\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{\boldsymbol{2}} \tag{2}$$

where V is the velocity vector, ρ the constant density, μ is the dynamic viscosity, α_1 and α_2 are the material constants and A_1 and A_2 are the first and second Rivlin-Ericksen tensors defined as

$$\tilde{\boldsymbol{A}}_{\mathbf{1}} = \tilde{\nabla}\tilde{\boldsymbol{V}} + \left(\tilde{\nabla}\tilde{\boldsymbol{V}}\right)^T \tag{3}$$

and

$$\tilde{\boldsymbol{A}}_{\mathbf{2}} = \frac{d\tilde{A}_1}{dt} + \tilde{\boldsymbol{A}}_{\mathbf{1}}(\tilde{\nabla}\tilde{\boldsymbol{V}}) + (\tilde{\boldsymbol{A}}_{\mathbf{1}}(\tilde{\nabla}\tilde{\boldsymbol{V}}))^T \tag{4}$$

For the model (1) required to be compatible with thermodynamics in the sense that all motions satisfy the Clasius-Duhem inequality and assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material parameters must meet the following conditions (Dunn and Fosdick (1974), Dunn and Rajagopal (1995)).

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \text{ and }\ \alpha_1 + \alpha_2 = 0 \tag{5}$$

3. PROBLEM FORMULATION

We consider an incompressible steady and laminar flow of a second grade fluid in a constricted tube of an infinite length having cosine shaped symmetric constriction of height δ. The radius of the unobstructed tube is R_0 and $R(\tilde{z})$ is the variable radius of the obstructed tube. The $\tilde{z}$ −axis is taken along the flow direction and $\tilde{r}$ −axis normal to it. Following (Forrester and Young (1970)) the boundary of the tube is taken as

$$R(\tilde{z}) = \begin{cases} R_0 - \dfrac{\delta}{2}\left(1 + \cos\left(\dfrac{\pi\tilde{z}}{z_0}\right)\right), & -z_0 < \tilde{z} < z_0 \\ R_0. & otherwise \end{cases} \tag{6}$$

In Eq. (6) z_0 is the length of the constricted region as shown in the Fig. 1. For steady axisymmetric

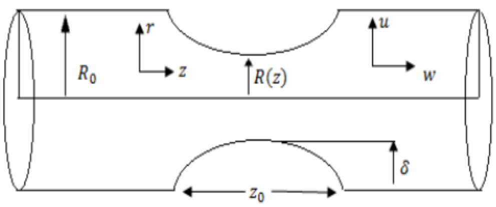

Fig. 1. Geometry of the problem.

flow of blood in tube, the velocity vector **V** is assumed to be of the form

$$\text{V} = \left[\tilde{u}(\tilde{r},\tilde{z}), 0, \tilde{w}(\tilde{r},\tilde{z})\right] \tag{7}$$

where $\tilde{u}$ and $\tilde{w}$ are the velocity components in $\tilde{r}$ −, $\tilde{z}$ −directions respectively. According to the geometry of the problem the boundary conditions are

$$\begin{aligned} &\tilde{u} = \tilde{w} = 0 \quad &&\text{at } \tilde{r} = R(\tilde{z}) \\ &\frac{\partial \tilde{w}}{\partial \tilde{r}} = 0 \quad &&\text{at } \tilde{r} = 0. \end{aligned} \tag{8}$$

In view of Eq. (8) the Eqs. (1) and (2) become

$$\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{9}$$

$$\begin{aligned} &\frac{\partial \tilde{h}}{\partial \tilde{r}} - \rho\tilde{w}\Omega = -\mu\frac{\partial \Omega}{\partial \tilde{z}} - \alpha_1\tilde{w}(\nabla^2\Omega - \frac{\Omega}{\tilde{r}^2}) \\ &+(\alpha_1 + \alpha_2)(\frac{2}{\tilde{r}}\frac{\partial(\tilde{u}\Omega)}{\partial \tilde{z}}), \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{\partial \tilde{h}}{\partial \tilde{z}} + \rho\tilde{u}\Omega = -\mu(\frac{\partial \Omega}{\partial \tilde{r}} + \frac{\Omega}{\partial \tilde{r}}) \\ &\alpha_1\tilde{u}(\nabla^2\Omega - \frac{\Omega}{\tilde{r}^2}) - (\alpha_1 + \alpha_2)(\frac{2}{\tilde{r}}\frac{\partial(\tilde{u}\Omega)}{\partial \tilde{r}}), \end{aligned} \tag{11}$$

where

$$\Omega = \frac{\partial \tilde{w}}{\partial \tilde{r}} - \frac{\partial \tilde{u}}{\partial \tilde{z}} \tag{12}$$

$$\tilde{h} = \frac{\rho}{2}(\tilde{u}^2 + \tilde{w}^2) - \alpha_1\left(\tilde{u}\nabla^2\left(\tilde{u} - \frac{\tilde{u}}{\tilde{r}^2}\right)\right) \tag{13}$$

$$\begin{aligned} &\tilde{\text{A}}_1{}^2 = 4(\frac{\partial \tilde{u}}{\partial \tilde{r}})^2 + 4(\frac{\tilde{u}}{\tilde{r}})^2 + 4(\frac{\partial \tilde{w}}{\partial \tilde{z}})^2 \\ &+2(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}})^2, \end{aligned} \tag{14}$$

and ∇^2 is the Laplacian parameter, $\tilde{h}$ is the generalized pressure. Introducing the dimensionless variables

$$\begin{aligned} &r = \frac{\tilde{r}}{R_0}, z = \frac{\tilde{z}}{z_0}, w = \frac{\tilde{w}}{U_0}, u = \frac{\tilde{u}z_0}{U_0\delta}, \\ &h = \frac{\tilde{h}}{\rho U_0^2}, p = \frac{\tilde{p}}{\rho U_0^2}, R_e = \frac{U_0 R_0 \rho}{\mu}, \end{aligned} \tag{15}$$

where U is the centerline velocity. An order-of-magnitude analysis is used to determine the negligible effect which appear in Eqs. (9) - (13). Now Eq. (9) becomes

$$\frac{\partial w}{\partial z} + \frac{\delta}{R_0}\frac{1}{r} \tag{16}$$

From Eq. (15) using order of magnitude analysis, which is also applicable for non-Newtonian fluids (Young (1968)), it is noted that $\frac{\partial w}{\partial z}$ is an order of, $\frac{\delta}{R_0}$ i.e. $\frac{\partial w}{\partial z} \sim o(\frac{\delta}{R_0})$. Forrester and Young (1970) assumed that for mild constriction if $1/R_e\delta/R_0 \ll 1$, $\delta/Z_0 \ll 1$ and $R_0/z_0 \approx 1$ then axial normal stress gradient $\frac{\partial^2 w}{\partial z^2}$ is negligible as compared to the gradient of shear component. So Eqs. (9) and (13) will become

$$\frac{\partial h}{\partial r} = 0. \tag{17}$$

$$\frac{\partial h}{\partial z} = \frac{1}{R_e}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{18}$$

$$h = w^2 - \alpha^* w\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \alpha^* \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + p. \tag{19}$$

where $\alpha^* = \frac{\alpha_1}{R_0^2 \rho}$. The non-dimensional form of co-sine shape constriction profile is

$$R(\tilde{z}) = \begin{cases} 1 - \frac{\delta^*}{2}\left(1+\cos(\pi z)\right), & -1 < z < 1 \\ 1. & otherwise \end{cases} \tag{20}$$

where $\delta^* = \delta / R_0$. Eq. (18) can be integrated from $r = 0$ to $r = R$ to get

$$\int_0^R r\frac{\partial h}{\partial z}dr = \frac{R}{R_e}\left(\frac{\partial w}{\partial r}\right)_R, \tag{21}$$

Exact solution of Eq. (21) can not be obtained. In order to find the approximate solution we assume fourth order polynomial which is called Karman-Pohlhausen approach Schlichting (1968). There-fore

$$\frac{w}{U} = A + B\left(1-\frac{r}{R}\right) + C\left(1-\frac{r}{R}\right)^2 + D\left(1-\frac{r}{R}\right)^3 + E\left(1-\frac{r}{R}\right)^4 \tag{22}$$

where U is the centerline velocity and A, B, C, D and E are undetermined coefficients which can be evaluated from the following five conditions

$$w = 0 \quad \text{at} \quad r = R, \tag{23}$$

$$w = U \quad \text{at} \quad r = 0, \tag{24}$$

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = 0, \tag{25}$$

$$\frac{\partial h}{\partial z} = \frac{1}{R_e}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = 0 \quad \text{at} \quad r = R, \tag{26}$$

$$\frac{\partial^2 w}{\partial r^2} = -2\frac{U}{R^2} \quad \text{at} \quad r = 0, \tag{27}$$

The no slip boundary conditions of zero velocity at the wall and centerline velocity U are given by Eqs. (23) and (24), condition (24) is a simple definition, condition (26) is obtained from Eq. (18). It is assumed that at $r = 0$ the velocity profile is parabolic at the center of the tube $\left(U\left[1-\frac{r^2}{R^2}\right]\right)$, so that the second derivative of w with respect to r, we get the condition (26). Thus Eq. (22) becomes

$$w = U\left[\left(\frac{-\lambda+10}{7}\right)\left(1-\frac{r}{R}\right) + \left(\frac{3\lambda+5}{7}\right)\left(1-\frac{r}{R}\right)^2 + \left(\frac{-3\lambda-12}{7}\right)\left(1-\frac{r}{R}\right)^3 + \left(\frac{\lambda+4}{7}\right)\left(1-\frac{r}{R}\right)^4\right] \tag{28}$$

where

$$\lambda = \frac{R^2 R_e}{U}\frac{dh}{dz} \tag{29}$$

We note that λ is the function of z only, since R,U and h depend only on z. In Eq. (29) U and h are unknowns. If Q is the flux through the tube, then

$$\int_0^R 2\pi r w dr. \tag{30}$$

Using Eq. (28) in (30), we obtain

$$Q = \frac{\pi R^2 U}{210}(-2\lambda U + 97U), \tag{31}$$

and centerline velocity U can also be written as

$$U = \frac{210}{97}.\frac{1}{\pi R^2}\left[Q + \frac{\pi R^4 R_e}{105}\frac{dh}{dz}\right], \tag{32}$$

Using Eq. (19) in (21),

$$\frac{1}{2}\frac{d}{dz}\int_0^R r w^2 dr + \frac{\alpha^*}{2}\frac{d}{dz}\int_0^R r\left(\frac{\partial w}{\partial r}\right)^2 dr + \frac{R^2}{2}\frac{dp}{dz} = \frac{1}{R_e}\left(\frac{\partial w}{\partial r}\right)_R, \tag{33}$$

and in order to obtain a closed solution one more approximation is taken into account that the velocity profile is parabolic

$$w = U\left[1 - \frac{r^2}{R^2}\right], \tag{34}$$

as discussed by Forrester and Young (1970). If we neglect the nonlinear terms, the flow will be a Poiseuille flow through the constriction (Young (1968)). Substitution of Eqs. (34) and (29 into Eq.(19) and (33) yields generalized pressure and pressure gradient

$$\frac{dh}{dz} = -48\alpha^* \frac{Q^2}{\pi^2}\frac{1}{R^7}\frac{dR}{dz} + \frac{496}{75}\alpha^* \frac{Q^2}{\pi^2}\frac{1}{R^5}\frac{dR}{dz} + \frac{388}{225}\frac{1}{R^5}\frac{dR}{dz} - \frac{8}{R^4 R_e}, \tag{35}$$

$$\frac{dp}{dz} = \frac{496}{75}\frac{Q^2}{\pi^2}\frac{\alpha^*}{R^7}\frac{dR}{dz} + \frac{388}{225}\frac{1}{R^5}\frac{Q}{\pi}\frac{dR}{dz} - \frac{8}{R^4 R_e}. \tag{36}$$

In order to obtain velocity w we substitute Eqs. (32) and (33) in Eq. (29) and (28), to get

$$w = \frac{2}{R^2}\frac{Q}{\pi}[2\eta\text{-}\eta^2] + \frac{4}{225}\frac{R_e}{R^3}\frac{Q^2}{\pi^2}\frac{dR}{dz}[-11\eta + 43\eta^2 - 45\eta^3 + 15\eta^4] + \frac{\alpha^*}{50925}\frac{Q^2}{\pi^2}\frac{R_e}{R^5}\frac{dR}{dz}[-58032\eta + 139376\eta^2 + 45687\eta^3 + 44144\eta^4] + \frac{\alpha^*}{97}\frac{Q^2}{\pi^2}\frac{R_e}{R^5}\frac{dR}{dz}[528\eta - 2064\eta^2 + 2160\eta^3 - 720\eta^4], \tag{37}$$

where $\eta = 1 - r/R$, is the velocity as a function of r and z through constricted tube. We can get velocity of unobstructed tube by taking R as constant or unity. The volume flow flux in unobstructed tube is

$Q = \pi R_0^2 U_0$, which gives nondimensional volume flux $Q = Q/R_0^2 U_0 = \pi$ whish is same for obstructed tube. Hence velocity w and pressure gradient $\frac{dp}{dz}$ will becomes

$$w = \frac{2}{R^2}[2\eta\text{-}\eta^2] + \frac{4}{225}\frac{R_e}{R^3}\frac{Q^2}{\pi^2}\frac{dR}{dz}[-11\eta + 43\eta^2 - 45\eta^3 + 15\eta^4] + \frac{\alpha^*}{50925}\frac{R_e}{R^5}\frac{dR}{dz}[-58032\eta + 139376\eta^2 + 45687\eta^3 + 44144\eta^4] + \frac{\alpha^*}{97}\frac{R_e}{R^5}\frac{dR}{dz}[528\eta - 2064\eta^2 + 2160\eta^3 - 720\eta^4], \tag{38}$$

$$\frac{dp}{dz} = \frac{496}{75}\frac{\alpha^*}{R^5}\frac{dR}{dz} + \frac{388}{225}\frac{1}{R^5}\frac{dR}{dz} - \frac{8}{R^4 R_e}. \tag{39}$$

The velocity profile for Forrester and Young (1970) can readily be recovered as a special case by setting $\alpha^* = 0,$ in Eq. (38).

4. Pressure Drop Across The Constriction And Across The Whole Length Of The Tube

We can get the pressure distribution at any cross section z along the stenosis Eq. (39) is integrated using boundary condition that is $p = p_0$ at $z = z$

$$(\Delta p) = \frac{496}{75}\alpha^* \int_{R_0}^{R} \frac{1}{R^5} dR + \frac{388}{225}\int_{R_0}^{R}\frac{1}{R^5}dR - \frac{8}{R_e}\int_{z_0}^{z}\frac{1}{R^4}dz, \tag{40}$$

Or

$$(\Delta p) = (\frac{124}{75}\alpha^* + \frac{97}{225})(\frac{1}{R^4} - \frac{1}{R_0^4}) - \frac{16}{\pi^2 R_e}\int_0^{\pi}\frac{1}{[a - b\cos u]^4}du, \tag{41}$$

where

$$a = 1 - \frac{\delta^*}{2}, \quad b = \frac{\delta^*}{2}. \tag{42}$$

Now

$$\int_0^{\pi}\frac{1}{a - b\cos u}du = \pi(a^2 - b^2)^{-1/2} \tag{43}$$

Differentiating Eq. (43) thrice partially with respect to a, we get

$$\int_0^{\pi}\frac{1}{[a - b\cos u]^4}du = \pi a(a^2 + \frac{3}{2}b^2)(a^2 - b^2)^{-7/2} = \pi f(\delta^*), \tag{44}$$

where

$$f(\delta^*) = (1 - \frac{\delta^*}{2})(1 - \delta^* + \frac{5}{8}(\delta^*)^2)(1 - \delta^*)^{-7/2}, \tag{45}$$

so that

$$(\Delta p) = (\frac{124}{75}\alpha^* + \frac{97}{225})(\frac{1}{R^4} - \frac{1}{R_0^4}) - \frac{16 z_0}{R_e R_0^4} f(\delta^*). \tag{46}$$

When there is no constriction i-e $\delta = 0$ and $f\left(\frac{\delta}{h_0}\right) = 1$, the pressure drop across the normal tube is given by

$$(\Delta p)_p = -\frac{16 z_0}{R_e R_0^4}. \tag{47}$$

In the absence of constriction, flow become Poiseuille and the subscript P denotes Poiseuille flow. If *2L* is the length of the tube, then the expression for the pressure across the whole length of the constricted tube is

$$[\Delta p] = (\frac{124}{75}\alpha^* + \frac{97}{225})(\frac{1}{R^4} - \frac{1}{R_0^4}) + \frac{8}{R_e R_0^4}(2L - 2z_0) \tag{48}$$

In the absence of constriction, $z_0 = 0$ the expression for the pressure of the normal tube will become

$$[\Delta p]_p = \frac{16L}{R_e R_0^4}. \tag{49}$$

We note that Eqs. (46) and (48) includes the results of Forrester and Young (1970) as a special case for $\alpha^* = 0$.

5. SHEAR STRESS ON CONSTRICTED SURFACE

The shear stress on the constricted surface is

$$\tau_{\tilde{w}} = \begin{pmatrix} \mu\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}}\right) \\ +\alpha_1\left(\tilde{u}\frac{\partial}{\partial \tilde{r}} + \tilde{w}\frac{\partial}{\partial \tilde{z}}\right)\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}}\right) \\ +2\frac{\partial \tilde{u}}{\partial \tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{z}} + 2\frac{\partial \tilde{w}}{\partial \tilde{r}}\frac{\partial \tilde{w}}{\partial \tilde{z}} \\ -\alpha_2\left[\frac{\tilde{u}}{\tilde{r}^2}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}}\right)\right] \end{pmatrix}_R . \tag{50}$$

Using Eq. (15) in Eq. (50), wall shearing stress becomes

$$\frac{\tau_{\tilde{w}}}{\rho U_0^2} = \frac{-1}{R_e}\left(\frac{\partial w}{\partial r}\right)_R - \alpha^*\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)_R . \tag{51}$$

From Eq. (34) and (51), we obtain

$$\tau_w = [\frac{4}{R^3} + \frac{R_e}{R^6}\frac{dR}{dz}(\frac{528}{97}\alpha^* - \frac{19344}{16975}\alpha^* R^2 - \frac{44}{225}R^2)][\frac{1}{R_e} + \alpha^*\frac{dR}{dz}(\frac{4}{R^3} + \frac{R_e}{R^6}\frac{dR}{dz}(\frac{528}{97}\alpha^* - \frac{19344}{16975}\alpha^* R^2 - \frac{44}{225}R^2))] = 0, \tag{52}$$

For $\alpha^* = 0$, the results of Forrester and Young (1970) for shear stress are recovered. Shear Stress in unobstructed tube will be

$$(\tau_w)_p = \frac{4}{R^3 R_e}. \tag{53}$$

At the middle of the constriction maximum value of shear stress occurs and the minimum value occurs at the ends of the constriction, their ratio can be expressed as

$$\frac{(\tau_w)_{\max}}{(\tau_w)_{\min}} = \frac{4}{(1-\delta^*)^3}. \tag{54}$$

The ratio given in Eq. (54) increases rapidly as constriction increases in size, as tabulated in Table 1.

Table 1 Effect of constriction on shear stress ratio

δ^*	0.1	0.2	0.3	0.4	0.5
$\frac{(\tau_w)_{max}}{(\tau_w)_{min}}$	1.0234	1.562	2.040	2.777	4.0

The present study holds only for mild stenosis, which is evident from the Table 1 that the stress at $\delta^* = 0.5$ (in the middle of the constriction) is four times the stress at the ends.

6. SEPARATION AND REATTACHMENT

Prandtl (1928) has explained the phenomena of separation in such a manner that the velocity of the fluid in the boundary layer drubbed towards the wall and inside the boundary layer the kinetic energy of the fluid particles appears to be less than that at the outer edge of the boundary layer. This means that the fluid particles inside the boundary layer may not be able to get the pressure which is applied in the outer layer. Even a small rise in pressure may trigger the fluid particles near at the wall to stop and turn back to form a recirculating flow region, which is the characteristic of the separated flows.

The separation and reattachment points can be calculated by taking negligible effects of shear stress at the wall, i.e $\tau_w = 0$.

$$[\frac{4}{R^3} + \frac{R_e}{R^6}\frac{dR}{dz}(\frac{528}{97}\alpha^* - \frac{19344}{16975}\alpha^* R^2 - \frac{44}{225}R^2)][\frac{1}{R_e} + \alpha^*\frac{dR}{dz}(\frac{4}{R^3} + \frac{R_e}{R^6}\frac{dR}{dz}(\frac{528}{97}\alpha^* - \frac{19344}{16975}\alpha^* R^2 - \frac{44}{225}R^2))] = 0, \tag{55}$$

$$R_e = \frac{152775R^2}{\frac{dR}{dz}(-207900\alpha^* + 7469R^2 + 43524\alpha^* R^2)}, \frac{-A \pm \sqrt{B}}{C}, \tag{56}$$

where

$$A = 152775R^3\alpha^*\frac{dR}{dz}, \quad B = 679\alpha^*(\frac{dR}{dz})^2(-55125\alpha^* + 7469R^2 + 43524\alpha^* R^2), C = 2\alpha^*(\frac{dR}{dz})^2[-207900\alpha^* + 7469R^2 + 43524\alpha^* R^2]. \tag{57}$$

It is noted that when $\alpha^* = 0$, the results of (Forrester and Young (1970)) for separation and reattachment are recovered.

7. RESULTS AND DISCUSSION

We are considering two-dimensional axial flow of a second grade fluid as a blood flowing in a constricted tube of infinite length. This geometry, of course, is intended to simulate an arterial stenosis, and the results are applicable to mild stenosis. The flow is assumed to be steady, laminar and an incompressible. An approximate method is used to get the solution for the velocity, pressure drop across the constriction length, across the whole length of the tube and shear stress on the constricted surface. The effect of different flow parameters on the fluid flow are simulated with the help of graphs and proper discussion related to each graph is also provided. In Figure 2 the variation of non-Newtonian parameter

α^* on the velocity profile is described at z = 0.475 taking $R_e = 30, \delta^* = 0.083$. It is denoted that velocity decreases with an increase in non-Newtonian parameter which seems physically to be correct. It can be seen from Fig. 3 that with an increase in Reynolds number velocity of the fluid decreases near the throat of the stenosis, however, it increases in the diverging region. Effect of Reynolds number for Newtonian fluids can be seen from Fig. 4, which is same as discussed in Young (1968), Forrester and Young (1970). The effect of Reynolds number on dimensionless pressure gradient between z = ±1 taking $\alpha^* = 0.05, \delta / R_0 = 0.1$ is shown in Fig. 5. It is well mentioned that the pressure gradient increases upto the throat of the constriction and then decreases in the diverging region.

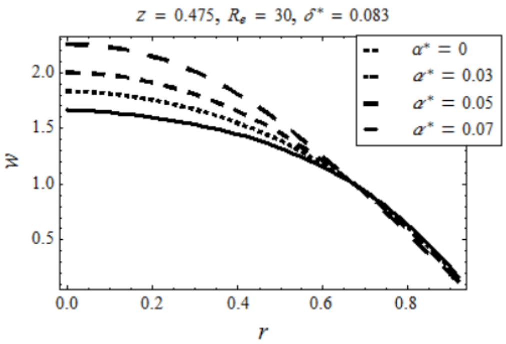

Fig. 2. Effect of non-Newtonian parameter α^* on velocity profile.

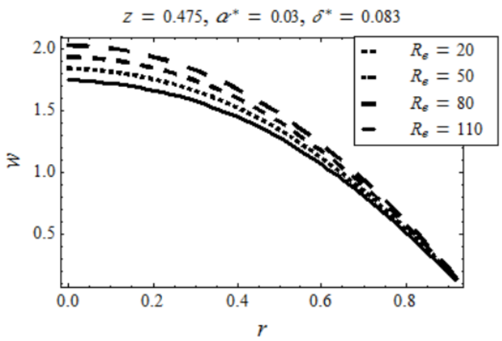

Fig. 3. Effect of Re on velocity profile for non-Newtonian fluid.

It is also observed that the value of pressure gradient at any point increases as Reynolds number increases in both converging and diverging region of constriction.

Same behavior of α^* and δ / R_0 on the pressure gradient is observed in Figs. 6 and 7.

It is evident from Figs. 8 and 9 that the results for Newtonain fluid are same as discussed in Young (1968), Forrester and Young (1970). The theoretical distribution of shearing stress along the wall is illustrated in Figs 10 − 14.

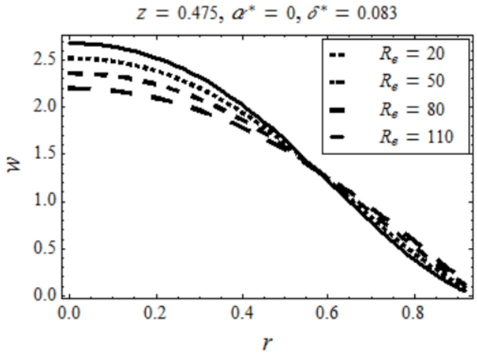

Fig. 4. Effect of R_e on velocity profile for Newtonian fluid.

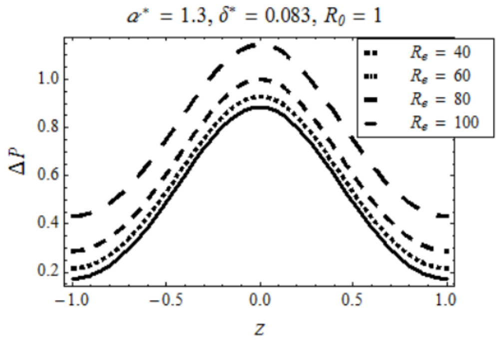

Fig. 5. Effect of R_e on pressure gradient for non-Newtonian fluid.

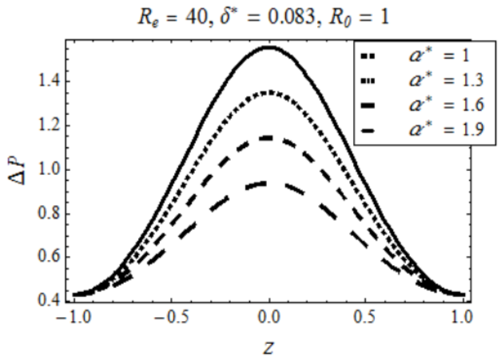

Fig. 6. Effect of non-Newtonian parameter α^* on pressure gradient for non-Newtonian fluid.

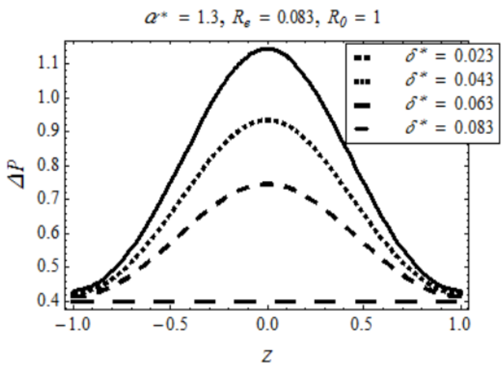

Fig. 7. Effect of δ^* on pressure gradient for non-Newtonian fluid.

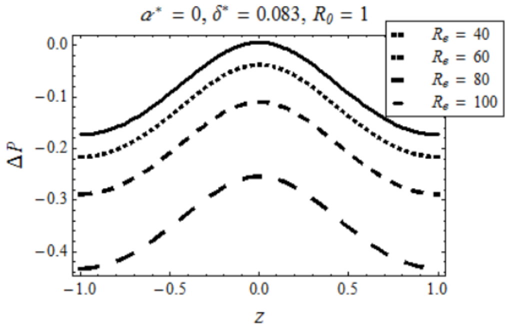

Fig. 8. Effect of R_e on pressure gradient for Newtonian fluid.

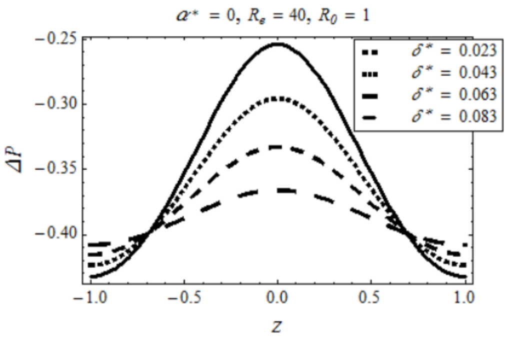

Fig. 9. Effect of δ∗ on pressure gradient for Newtonian fluid.

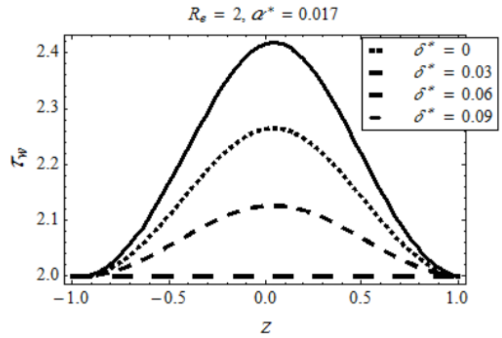

Fig. 10. Effect of δ^* on shear stress for non-Newtonian fluid.

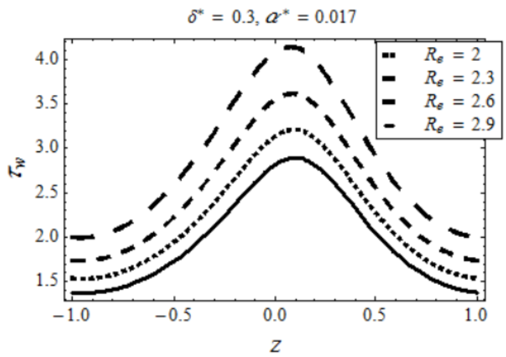

Fig. 11. Effect of R_e on shear stress for non-Newtonian fluid.

Fig. 10 depicts the influence of constriction height on wall shear stress. It is noted that with an increase in constriction height δ^* wall shear stress increases, and its maximum value occurs at the middle of the constriction, which seems physically to be correct. It is also observed that in the absence of constriction, the flow is fully developed or Poiseuille flow. It is observed from the Fig. 11 that for any Reynolds number, the shearing stress reaches a maximum value on the throat and then rapidly decreases in the diverging region. It is also noted that shear stress decreases with an increase in Reynolds number. It means that Reynolds number provides a mechanism to control the wall shear stress. From Fig. 12, it is well mentioning, as expected, that as non-Newtonian parameter α∗ increases wall shear stress also increases.

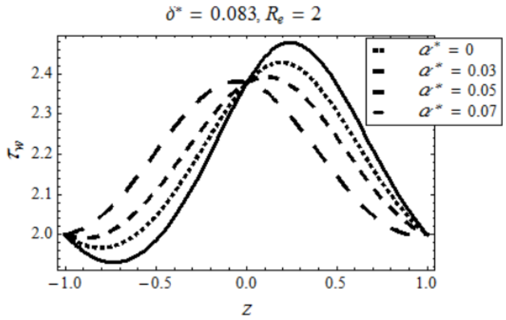

Fig. 12. Effect of non-Newtonian parameter α^* on shear stress.

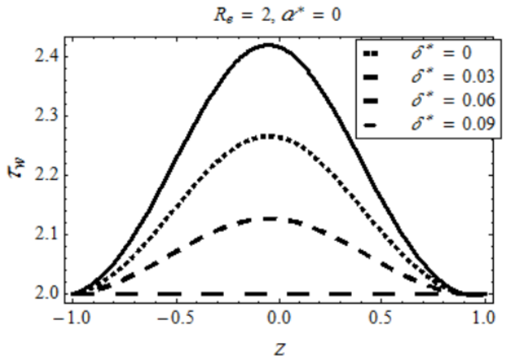

Fig. 13. Effect of δ∗ on shear stress for Newtonian fluid.

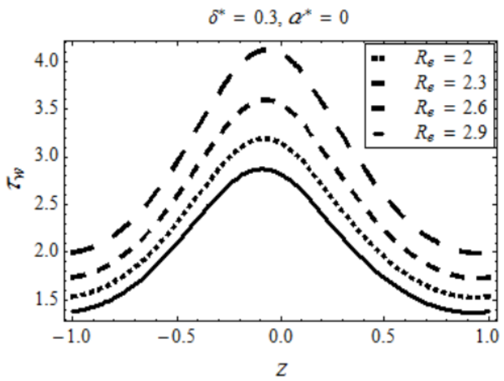

Fig. 14. Effect of Re on shear stress for non-Newtonian fluid.

As a special case (Young (1968), Forrester and Young (1970)) for Newtonian fluid can be seen in

Figs. 13 and 14. Figs. 15 and 16 give the influence of constriction on the separation and reattachment points respectively. It is observed, as naturally expected, that separation point moves upstream with an increase in constriction size while reattachment point moves downstream. It is also observed that the separation point moves upstream and the reattachment point moves downstream as the Reynolds number increases.

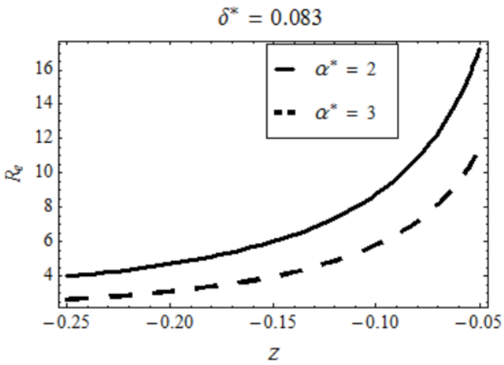

Fig. 15. Separation point for α∗ in converging region.

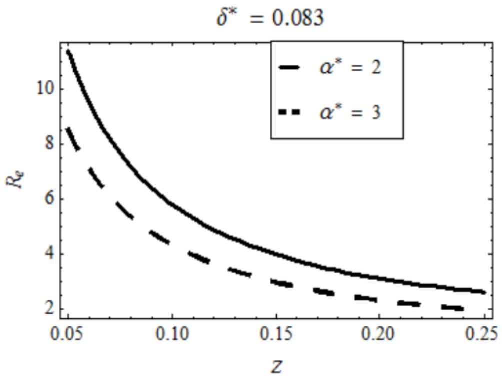

Fig. 16. Reattachment point α^* in diverging region.

8. CONCLUSION

In the present study an incompressible laminar and steady flow of a second grade fluid through constricted tube is modeled and analyzed theoretically. The fluid is assumed to be blood flowing through the artery and the results are applicable to mild stenosis. The expressions for velocity field, pressure gradient, wall shear stress and separation phenomena for the geometry of the constriction are presented. An integral momentum method is applied for the solution of the problem. In human body blood flow is laminar so the Reynold number taken in the present theoretical study is very close to natural phenomena (Young (1968), Forrester and Young (1970)). The summary of findings of the present work is as follows :

- Velocity decreases with an increase in non-Newtonian parameter
- Viscous forces are dominant over inertia forces near the throat of the constriction, however, opposite effect is observed in the diverging region
- Reynolds number and non-Newtonian parameter are economical parameters to control the wall shear stress
- Reynolds number also provides a mechanism to control the separation and reattachment points
- The separation and reattachment points strongly depend upon constriction height
- The present study includes the theoretical and experimental results for the velocity profile, pressure gradient and wall shear stress of (Forrester and Young 1974) as a special case for α∗ = 0

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