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Optimum Geometric Bifurcation under Pulsating Flow Assuming Minimum Energy Consumption in Cardiovascular System, an Extension on Murray's Law

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ABSTRACT

In a bifurcation including a mother artery and two daughter arteries, the energy drop is minimum, if, the cube of the radius of the mother artery equals the sum of the cube of the radii of daughter arteries. This is the expression of Murray's law (or cubic law) assuming the flow is steady. In this paper, an extension of Murray's law is investigated using the minimum energy hypothesis, totally analytical for pulsating flow. In addition to the two terms that Murray considered in his calculations, there is additional energy to move fluid toward and back in the pulsating flow. This additional energy is calculated and added to two other parts of energy in Murray's analysis, and then optimized. The relationships for diameters and the angle between daughter arteries are extended. The effect of frequency and Womersley number have appeared as coefficients in the relations. According to the results, the most difference between Murray's law for both diameters and the angle between daughter arteries, and the relationship derived in the present paper, occurs in Womersley number between 2 and 5. For a special case which in the daughter arteries have the same diameter, the power of diameters varies up from 3 to 3.2. Also, for this special case, there is maximum 6 degrees difference with Murray's law for the angle between daughter arteries. In short, the obtained relations, assuming pulsating flow, do not yield very different results from Murray's law assuming steady flow.

Keywords: Cubic law; Minimum energy hypothesis; Arterial junctions; Oscillatory flow; Womersley number; Optimization.

NOMENCLUCTURE

W	Womersley number	μ	dynamic viscosity
C	steady pressure gradient	ω	angular frequency
T	a cycle of oscillation	ρ	density
p	pressure	u_s	steady component of velocity
R	radius of artery	u_o	oscillating component of velocity
J_0	first kind and zeroth order Bessel function	i	imaginary unit
J_1	first kind and first order Bessel function	l	length of artery
Q_s	steady flow rate	E_{tot}	total energy per a cycle
Q_o	oscillating flow rate	α, β	diameter ratios of daughter arteries to mother artery
x, y	angles between daughter arteries	f, A_0, A_1, A_2	two variables function of Womersley number and diameter ratios, the coefficients of radii in expanded relationships

1. INTRODUCTION

There has always been an important and undeniable relationship between the shape and geometry of a system and its function. This fact has been shown experimentally again and again throughout the years. The cardiovascular system in men and animals is not an exception to this rule. On the other hand, one of the interests of theoretical physiology is to find the governing relations on systems such as the cardiovascular system (Murray 1926a). The cardiovascular system has sensitive points which are susceptible to diseases that could be leading to death. For example, fat plaques mostly gather in places such as bifurcations and curvature of arteries, and it is one of the main Atherosclerosis occurrence reasons. Thus, it seems reasonable to study the hemodynamics in such places (Fernandez *et al.* 1976). The main duty of cardiovascular and blood flow is to feed cells nutrients and oxygen and to remove waste materials from them. To better understand and investigate, knowing the behavior of blood flow in some important places like the heart or coronary arteries in both healthy and sick people, can provide useful insights into the natural function of the cardiovascular and how the function changes after getting sick. Cardiovascular is very delicate and sensitive so that a little change in the shape and anatomy caused by illnesses could have significant influences on blood flow and hemodynamics. Pressure distribution, velocity, and shear stress are some of the important properties of hemodynamics. As mentioned previously, arterial bifurcations are very sensitive points in the cardiovascular system and it is clear that they are playing a vital role in the body of men and other mammals. Because the heart can transfer blood to millions of cells by arterial bifurcations in the form that, an artery divides into two other arteries and each one of these two bifurcated arteries divides into two arteries itself, and it goes on until all cells are fed by the blood flow. As there are millions of cells in the body, so there are millions of arterial bifurcations too, which is a remarkable number. So in such a complicated system any problem can even lead to death and should be taken care of. For example, the circle of Willis in the brain and coronary bifurcations of the heart could be mentioned as some of the most important places in the cardiovascular system which are susceptible to diseases like Aneurysm or Atherosclerosis. Any problem in such places can lead to stroke, heart attack, and disability (Akay 2006). Thus, finding out the effective factors in such systems can help to a better understanding and prevent some serious illnesses. Murray (1926a, b) showed that there is a relation between flow rate and volume of an artery in the form that, with steady-state assumption, the flow rate in an artery is proportional with the cube of the radius according to the minimum energy hypothesis. So due to the conservation of mass and minimum energy hypothesis, the relationship for a mother artery that divides into two daughter arteries is $R_1^3 = R_2^3 + R_3^3$. In this relationship, R_1, R_2 and R_3 are the radii of the mother artery and two daughter arteries respectively. Similarly, Murray derived a relationship for the optimum angle between two daughter arteries and in a special case which in the daughter arteries have the same diameter, this angle is almost 75 degrees. Fernandez *et al.* (1976) while pointing to the importance of the reason for illnesses and the fact that fiber plaques mostly gather in arterial bifurcations, investigated pulsating flow in a bifurcation numerically and explained some important results for Atherogenesis. Zamir (1976) introduced a new principle called minimum drag and calculated the optimum angle between daughter arteries by using that. Zamir (1978) also investigated asymmetric bifurcations, and using four different optimum principles showed that there is not much difference between them. Sherman (1981) derived Murray's law by minimization of the resistance of the bifurcation and stated that the conservation of Murray's law depends on being the resistance of bifurcation minimal. Bejan *et al.* (2000) showed that the maximum thermodynamic operation of pure fluids depends on minimization of resistance or minimization of entropy generation when the flow rate is constant. They also mentioned the structure of geometry plays a significant role in maximum performance. Taber *et al.* (2001) by studying almost 450 capillaries in a chicken embryo found out that Murray's law has good accommodation with their measurements. Kasab (2006) based on his studies found out the design of arterial trees depends on minimization of resistance, and also arterial trees obey from minimum energy hypothesis in various organizations. Painter *et al.* (2006) to extend Murray's law in the cardiovascular system solved pulsating flow in a rigid and elastic tube and presented an exact solution for velocity and shear force. They also showed that in small arteries and Womersley numbers, their relationship leads to Murray's law. Revellin *et al.* (2009) with the assumption that the non-Newtonian fluid for blood is more appropriate, generated Murray's law under two constant volume and constant surface constraints for steady flow. With constant volume, they got to the same Murray's law but vice versa with constant surface Murray's relationship changes so that the power of diameters changes between 2.42 and 3 and its value depends on applied constraint and properties of the fluid. For the Newtonian model, the power is 2.5. Bui *et al.* (2009) developed a CFD model for pulsating flow in an arterial tree model of arteries in the brain, considered a computer model containing a fractal model of vessels, and a describing mathematical model for blood transfer. Lee and Lee (2010) investigated 140 capillaries of a three-day-old chicken embryo and concluded there is a good accommodation between Murray's law and the diameters they measured. Also, the average angle between daughter arteries was 77 degrees which almost agrees with the angle of 75 degrees derived by Murray's law. Huo *et al.* (2009, 2012) by using the minimum energy hypothesis, derived new relationships analytically which the power of the diameters in them was different from the power of diameters in Murray's law. Then, they investigated the bifurcations accommodated on their relationships and the bifurcations based on Murray's law numerically. Matsuo *et al.* (2013) measured the size of vessels in the brain, kidney, and earlobe of a cat and a chicken embryo in bifurcated points and tried to determine the power of diameters in Murray's law. Silva and Reis (2014) based on minimizing total impedance, generalized the scaling relationships for pulsating flow. They showed that in the geometry of arterial tree under constant volume constraint when frequency tends to zero, their relationship leads to Murray's law. They also showed that the optimum scaling depends on frequency, the flexibility of walls, and the symmetry of daughter arteries. Miguel (2016) investigated the laminar Newtonian and power-law model as a non-Newtonian model in both symmetric and asymmetric bifurcations and derived relationships between the mother and daughter arteries. Golzar *et al.* (2017) studied the optimum distribution of wall shear stress and its influences on the morphology of blood arteries. They showed that the wall shear stress in laminar steady flow is a constant value while in turbulent flow it has an exponential relation with the diameter in the form that the power of the diameter changes with the roughness of the tube. They derived relationships between mother and daughter arteries based on the optimum distribution of shear stress. They also showed that the optimum distribution of shear stress minimizes the resistance just like the minimum energy hypothesis. Srinivasacharya and Rao (2018) studied pulsating blood flow as stress coupled non-Newtonian model in a bifurcation with mild tightness in the mother artery by finite volume method. They concluded that the flow rate increases when the Womersley number and angle between the daughter arteries decreases, also shear stress decreases when the Womersley number and the angle between daughters increases. Miguel (2018) presented an analytical solution for fluid flow and heat transfer in the optimum bifurcation of tubes. He concluded that the structural features of such networks depend on the ratios of diameters and the angle between daughter arteries. Xu and Chen (2019) designed a Y-shaped micromixer by using generalized Murray's law and this micromixer had a 90 percent efficiency. Kashyap *et al.* (2020) investigated the effect of curvature on bifurcations on Atherosclerosis using an ideal model of coronary artery bifurcation studied the hemodynamic indexes such as wall shear stress and oscillatory shear index. Sciubba (2020) emphasizing the fact that the exergy cost is an accurate and scientific method for the geometry of bifurcations in engineering applications, used exergy cost for developing the structure of bifurcations. Rosenberg (2021) stated that however Sherman's work in 1981 is true but it is flawed so that, when an artery divides into N equal arteries there is a special number of vessel radii which abbeys Murray's law but the resistance is not necessarily minimum in all the cases. He also proved that the value of minimum resistance increases with an increase in the number of N daughter bifurcated arteries.

2. PROBLEM DESCRIPTION

A large number of scientists have tried to analyze the blood flow since almost a hundred years ago. In the beginning, investigating and studies were simplified mathematical models like the analytical solution of pulsating flow in a single tube derived by Womersley. In such models, an artery is usually considered a rigid tube with a circular cross-section. That is one of the most important assumptions because it would significantly help to simplify the governing equations. One of the other assumptions has used, is that the blood flow is laminar and fully developed. The fully developed assumption turns the momentum equation from a non-linear equation to a linear equation and makes the solution easier. We know that the operation of the heart is in the form that produces a pulsating pressure gradient in time. An analytical solution with applying this complicated pressure gradient in the momentum equation is almost impossible. But according to the fully developed assumption, the momentum equation is a linear equation which means, it obeys from superposition principle. That is if in a special case the pressure is p_1 and the velocity is u_1 and in another case the pressure is p_2 and the velocity is u_2, then for the pressure $p_1 + p_2$ the velocity is $u_1 + u_2$. On the other hand, we know that any periodic function can be written as an infinitive summation of sine and cosine terms known as the Fourier series. Therefore, using the Fourier series instead of the complicated pressure gradient produced by the heart is reasonable (Zamir 2000). Because we can solve momentum equation for any sine or cosine pressure gradient separately then sum the answers and it is much easier. In the present paper, we assume that the axial pressure gradient is $\partial p/\partial z = C + Csin(\omega t)$ which means, the pressure gradient in the momentum equation consists of two parts. ω is the radial frequency and it is equal to $2\pi/T$ which in T is a cycle of time. The first part is a constant value and the second part is a sine as oscillatory part. The reason for the existence of constant part is the fact that blood flows in the body, so there should be a part in the pressure gradient to move the fluid toward and makes a net flow. According to the said cases, it seems reasonable that there should be an additional part that needs more energy to create such flow, however makes no net flow in a cycle. Physically, because of the sine part in the pressure gradient, the fluid is moved toward then it is backed to the first place and in a cycle, it makes no net flow. But moving fluid in such a way needs energy itself. The strategy we follow in this paper is to calculate this extra term and adding it to two other parts in Murray's analysis and finally optimize this new expression.

3. MATHEMATICAL ANALYSIS

The momentum equation for laminar, Newtonian, and fully developed flow in a rigid tube with a circular section is:

$$\rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

If the pressure gradient is $\partial p/\partial z = C + Csin(\omega t)$ as explained before, then because Eq. (1) obeys superposition we have:

$$\left(C-\mu\left(\frac{d^2 u_s}{dr^2}+\frac{1}{r}\frac{du_s}{dr}\right)\right) +\left(\rho\frac{\partial u_o}{\partial t}+Csin(\omega t)-\mu\left(\frac{\partial^2 u_o}{\partial r^2}+\frac{1}{r}\frac{\partial u_o}{\partial r}\right)\right)=0 \tag{2}$$

Equation (2) means that steady and unsteady parts are separated from each other. Noticing that the two groups above are different and their summation is zero, therefore, each one should be zero separately. The solution of the first part leads to Poiseuille flow and the second part is the governing equation on pure oscillatory flow. We pass up from Poiseuille solution because it is well known. For the solution of the oscillatory part, we reference Womersley's solution (Womersley 1955). Womersley used the superposition principle for solving the second part in Eq. (2), so that by choosing an intelligently linear combination of sine and cosine based on Euler's formula in complex analysis for pressure gradient so that the solution by separation of variables method would be much easier. Finally, the velocity field derived by solving the oscillatory governing equation is:

$$u_o(r,t) = \frac{iCR^2}{\mu W^2}\left(1 - \frac{J_0\left(\frac{r}{R} i^{1.5} W\right)}{J_0(i^{1.5}W)}\right) e^{i\omega t} \quad (3)$$

In Eq. (3) W is the Womersley number. Notice that there are two solutions in Eq. (3) at the same time. One of them corresponds to the sine pressure gradient and the other one corresponds to the cosine pressure gradient. We could also understand this point by paying attention to the existence of complex terms because we know that complex numbers don't have a physical meaning. They only simplified the solution and that's it. To calculate flow rate, we should integrate the velocity over a section then we have:

$$Q_o = \frac{\pi i C R^4}{\mu W^2}\left(1 - \frac{2J_1(i^{1.5}W)}{i^{1.5} W J_0(i^{1.5}W)}\right) e^{i\omega t} \quad (4)$$

There are two flow rates in Eq. (4) in the same way. One of them corresponds to the sine pressure gradient and the other one corresponds to the cosine pressure gradient. In the following we will continue the calculating with the sine pressure gradient, so we should consider the imaginary parts of Eqs. (3) and (4). We will use the following equation for calculating the energy drop for the oscillatory part of the flow:

$$P = -l.\Delta p/l.Q_o \quad (5)$$

In Eq. (5) P is the needed power to move the fluid and l is the length of the tube. Δp, is the pressure difference between the two ends of the tube. By replacing the pressure gradient and flow rate in Eq. (5) we have

$$P = -l.C\sin(\omega t).Im(Q_o) \quad (6)$$

By doing a little simplification we get to:

$$P = -lC\sin(\omega t).Im\left(\frac{Q_o}{Q_s}Q_s\right) = -lC\sin(\omega t)Q_s.Im\left(\frac{Q_o}{Q_s}\right) \rightarrow$$
$$P = -lC\sin(\omega t)Q_s.Im\left(\frac{-8}{i^3W^2}\left(1 - \frac{2J_1(i^{1.5}W)}{i^{1.5}WJ_0(i^{1.5}W)}\right)e^{i\omega t}\right) \quad (7)$$

We can see the most important point in these calculations, in Eq. (7). As we are dealing with two Bessel functions and one of them is divided on the other, and each one is an infinitive series with a complex argument, it seems a little hard to separate the imaginary part of the expression in the parentheses in Eq. (7) as we considered the pressure gradient in a sine form. It may seem hard at first look to separate the real part and imaginary part of such expression. But there is an important point here, as we are dealing with a complex expression, so we can use the polar form of complex numbers. We know that in the complex analysis if k is a complex number, then it can be written in two forms, Cartesian form, and polar form. These two forms can be connected by the following relation:

$$k = m + in = \sqrt{m^2+n^2}e^{itg^{-1}(m/n)} = |k|e^{i\theta} \quad (8)$$

So for further simplification and getting rid of Bessel functions, since the expression in parentheses in Eq. (7) is a complex expression, we can name it g, so we have $g = \frac{1}{i^3W^2}\left(1 - \frac{2J_1(i^{1.5}W)}{i^{1.5}WJ_0(i^{1.5}W)}\right)$ then according to Eq. (8) and Euler's formula we have:

$$g = |g|e^{i\theta_g} = |g|\left(\cos(\theta_g) + i\sin(\theta_g)\right) \quad (9)$$

g is intended to simplify calculations. And based on Eq. (8), subsequently, $|g|$ and θ_g are modulus and argument of g respectively. By replacing Eq. (9) into Eq. (7) and doing some simple mathematical calculations we have:

$$P = 8lC\sin(\omega t)Q_s.|g|\sin(\theta_g + \omega t) \quad (10)$$

We should consider two points in Eq. (10). First, the left-hand side of the equation is the power, not energy, and it is a function of time. Second, we know that for the flow rate Q_s there is a relationship coming below:

$$Q_s = \frac{-C\pi R^4}{8\mu} \quad (11)$$

Finally, by replacing Eq. (11) into Eq. (10) and then integrating over a cycle we get to the following equation:

$$E_o = -\frac{32\mu l}{\pi R^4}Q_s^2|g|\cos(\theta_g) \quad (12)$$

Equation (12) is an averaged energy over a cycle. This equation is the term we wanted to find and add to the two other terms in Murray's analysis and then optimize the new relation. About the negative sign in Eq. (12), we can say that it makes the entire right-hand side positive, so this ambiguity should not occur that the energy is negative in Eq. (12).

3.1 Generalized Relationship for Diameters

By adding the Eq. (12) in two other terms in Murray's analysis we have:

$$E_{tot} = \frac{8\mu l}{\pi R^4} Q_s^2 - \frac{32\mu l}{\pi R^4} Q_s^2 |g| \cos(\theta_g) + b\pi R^2 l \quad (13)$$

$$\frac{dE_{tot}}{dR} = \frac{d}{dR}\left(\frac{8\mu l}{\pi R^4} Q_s^2 \left(1 - 4|g|\cos(\theta_g)\right) + b\pi R^2 l\right)$$
$$= \frac{-32\mu l}{\pi R^5} Q_s^2 \left(1 - 4|g|\cos(\theta_g)\right) - \frac{32\mu l}{\pi R^4} Q_s^2 \frac{d}{dW}\left(|g|\cos(\theta_g)\right)\frac{dW}{dR} + 2b\pi R l$$
$$= \frac{-32\mu l}{\pi R^5} Q_s^2 \left(1 - 4|g|\cos(\theta_g)\right) - \frac{32\mu l}{\pi R^4} Q_s^2 \sqrt{\frac{\rho\omega}{\mu}} \frac{d}{dW}\left(|g|\cos(\theta_g)\right) + 2b\pi R l$$

Notice that in the two last lines of calculations, we used the chain rule in derivative because Eq. (12) is a function of Womersley number and Womersley number is a function of radius itself. If we multiply the second term on the right-hand side in then we have:

$$= \frac{-32\mu l}{\pi R^5} Q_s^2 \left(1 - 4|g|\cos(\theta_g)\right) - \frac{32\mu l}{\pi R^5} Q_s^2 R \sqrt{\frac{\rho\omega}{\mu}} \frac{d}{dW}\left(|g|\cos(\theta_g)\right) + 2b\pi R l$$

The term $R\sqrt{\frac{\rho\omega}{\mu}}$ is the definition of Womersley number, W. So the final and simplified form of the above expression which should be equal to zero would be like the following equation:

$$\frac{-32\mu l}{\pi R^5} Q_s^2 \left(1 - 4|g|\cos(\theta_g)\right) - \frac{32\mu l}{\pi R^5} Q_s^2 W \frac{d}{dW}\left(|g|\cos(\theta_g)\right) + 2b\pi R l = 0$$

$$\frac{32\mu l}{\pi R^5} Q_s^2 \left(1 - 4|g|\cos(\theta_g) + W\frac{d}{dW}\left(|g|\cos(\theta_g)\right)\right) = 2b\pi R l \rightarrow$$

$$Q_s^2 = \pi^2 b R^6 \Big/ \left(16\mu \left(1 - 4|g|\cos(\theta_g) + W\frac{d}{dW}\left(|g|\cos(\theta_g)\right)\right)\right) \quad (14)$$

$$Q_s = \sqrt{\frac{\pi^2 b}{16\mu}}\, R^3 \left(1 - 4|g|\cos(\theta_g) + W\frac{d}{dW}\left(|g|\cos(\theta_g)\right)\right)^{-0.5} \quad (15)$$

With this argument and method, we can connect the flow rate and Womersley number. An important point here, is, however, the net flow rate is independent of frequency and Womersley number, but the thing matters in Eq. (15) are that Womersley number is a function of radius and frequency. The frequency is almost a constant value as the heart beats at a constant number every minute. Therefore, we can say that the expression under radical can be a function of radius. Also notice that, if the frequency tends to zero the expression under the radical tends to 0.5 because it has averaged and if we use the conservation of mass we get to Murray's law, therefore it is reasonable that we say the proportion matters, not the equality. If we name the expression under the radical, according to the conservation of mass in a bifurcation shown in Fig. 1 we have:

$$Q_{mother\ 0} = Q_{daughter\ 1} + Q_{daughter\ 2} \quad (16)$$

And because $Q_s \propto \frac{R^3}{\sqrt{h(W)}}$ in which $h(W) = 1 - 4|g|\cos(\theta_g) + W\frac{d}{dW}(|g|\cos(\theta_g))$ so we have:

$$\frac{R_0^3}{\sqrt{h(W_0)}} = \frac{R_1^3}{\sqrt{h(W_1)}} + \frac{R_2^3}{\sqrt{h(W_2)}} \quad (17)$$

In Eq. (17), the terms under radicals are functions of the Womersley number. The Eq. (17) could be written in dimensionless form for better understanding:

$$f(\alpha, W_0)\alpha^3 + f(\beta, W_0)\beta^3 = 1 \quad (18)$$

Equation 18 is the dimensionless form of Eq. (17) which in $\alpha = R_1/R_0$ and $\beta = R_2/R_0$ are ratios of the diameters of daughter arteries to mother artery and W_0 is Womersley number in mother artery. Also, the coefficients are $f(\alpha, W_0) = \sqrt{h(W_0)/h(W_1)}$ and $f(\beta, W_0) = \sqrt{h(W_0)/h(W_2)}$. So the influence of frequency and Womersley number are in the form of coefficients of diameter ratios in Eq. 18. If we have one of the ratios of daughter to mother diameters and Womersley number in the mother artery then we can calculate the other ratio using Eq. (18). For better usage of Eq. (18), we can plot the coefficients according to Womersley number for different ratios (α or β). After choosing a special ratio and a Womersley number we should find the other ratio. To do this, $1 - f(\alpha, W_0)\alpha^3$ also is plotted according to the Womersley number and different ratios.

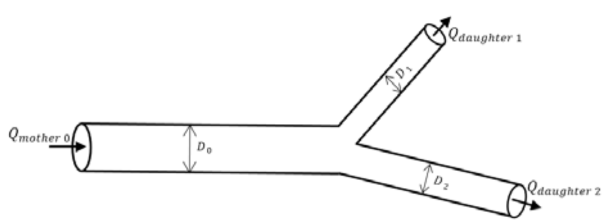

Fig. 1: A schematic Image of a simple bifurcation including a mother artery with diameter D_0 and two daughter arteries with diameters D_1 and D_2 respectively.

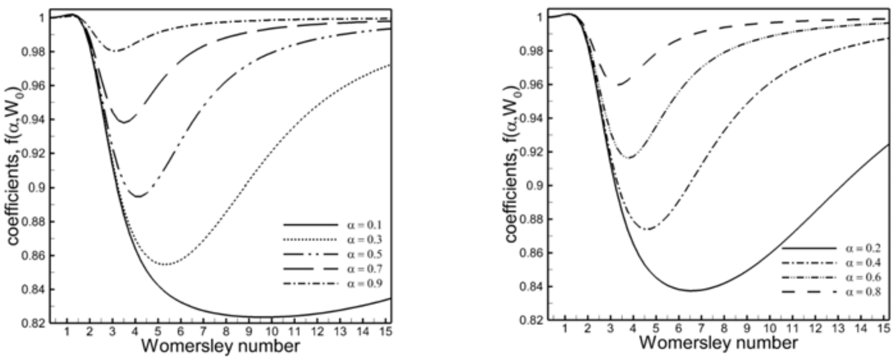

Fig. 2. Coefficients of Eq. 18 for different Womrsley numbers in mother artery and diameter ratio of daughter to mother's diameter, α.

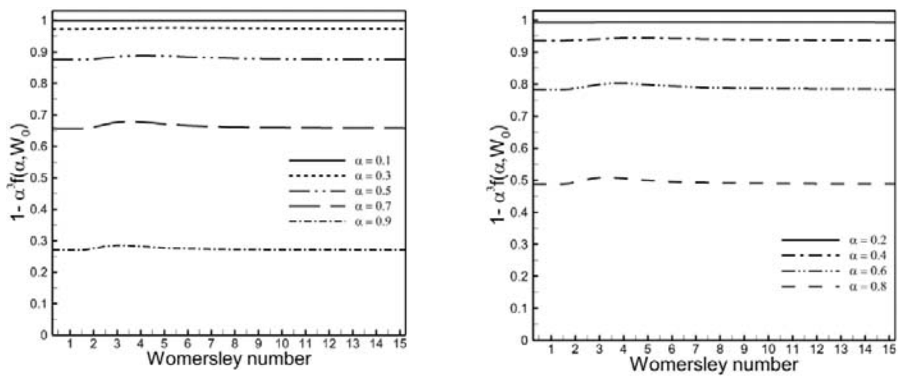

Fig. 3. These diagrams are for finding β with having α and W in mother artery.

Figure 2 shows us some important points. First, for all ratios from 0.1 to 0.9 when the Womersley number tends to zero all the values of coefficients tend to 1 which means, the relationship (18) turns to Murray's law. Physically, when the frequency of flow tends to zero, the flow turns to a steady flow, and as we know Murray's law is derived based on the steady flow assumption. Second, the diameter of daughter artery to diameter of mother artery ratio is smaller the variation of the corresponding coefficient is higher so that we can see almost 17 percent difference for ratio 0.1. Third, most variation in coefficients occurs between Womersley numbers 2 and 8 on average. Another interesting thing is the value of the ratio is a bigger number the variation of the coefficient is smaller. Another interesting thing we can find out from Fig. 2. is the Womersley number increases the values of almost all the ratios coefficients tends to 1 and the relationship (18) gets closer to Murray's law again. This is the physical justification for the fact that the Womersley number is higher the profile of the oscillatory part of velocity tends to a more flat profile with a highly smaller peak value. Therefore, the difference between pulsating flow and steady flow is less in high Womersley numbers The diagrams in Fig. 3 show an important result, and that is, however, the Eq. (18) is different with Murray's law and the coefficients change with Womersley number and have different values from 1, but the influence of frequency and Womersley number on the final answer which is the unknown ratio (β) only in the scope of 2 to 5 is not sensible. From a mathematical point of view, as the ratios are numbers between 0 and 1 and when they are put in the relationship (18) reach a power of 3, so they will be a very small value and lower the effect of the coefficients highly too. For a better comparison with Murray's law, we can say that the dimensionless form of Murray's law is, $\alpha^3 + \beta^3 = 1$ and for a special case which in the daughter arteries have the same radii, it can be written in the form $2\alpha^3 = 1$ or $\alpha = 2^{-\frac{1}{3}}$. Now if $\alpha = \beta$, in Eq. (18) then we have:

$$2f(\alpha, W_0)\alpha^3 = 1 \tag{19}$$

This Eq. (19) is an implicit equation relative to α. If we rewrite this equation as follows and solve it, then we can have a good comparison between the power of diameters in Murray's law and the power of the following Eq. (20):

$$\alpha^3 = 2^{-1} f(\alpha, W_0)^{-1} \ \rightarrow \ \alpha = 2^{\frac{-1}{3}} f(\alpha, W_0)^{\frac{-1}{3}}$$

$$\alpha = 2^{\frac{-1}{3}} \left(2^{log_2^{f(\alpha,W_0)}}\right)^{\frac{-1}{3}} = 2^{-\frac{1+log_2^{f(\alpha,W_0)}}{3}} = 2^{\frac{\frac{-1}{3}}{1+log_2^{f(\alpha,W_0)}}}$$

$$\alpha = 2^{\frac{\frac{-1}{3}}{1+log_2^{f(\alpha,W_0)}}} = 2^{\frac{-1}{m}} \tag{20}$$

If we plot the power m in Eq. (20) according to the Womersley number we have:

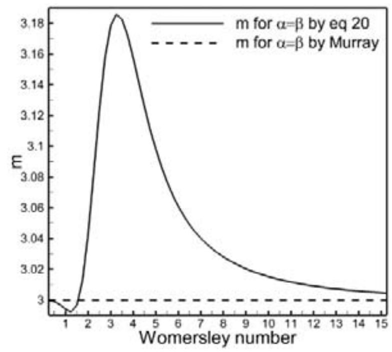

Fig. 4. Power of diameter, m, in Eq. 20 according to the Womersley number compared with corresponding power of diameter in Murray's law.

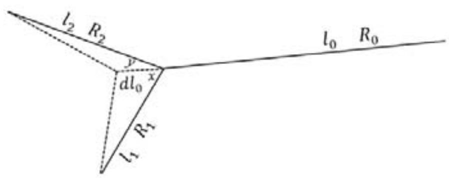

Fig. 5. A virtual length change along the mother artery, dl_0.

According to Fig. 4, the power in Eq. (20) varies up to about 3.2. On the other hand, as expected the most variation is between Womersley numbers 2 and 5. For numbers larger than 5, m tends to 3 and the reason is that the oscillatory part of the velocity profile is getting more flat as mentioned before.

Also, for small Womersley numbers tends to 3 because the flow tends to a steady flow.

3.2 Generalized Relationships for Angles

Murray in 1926, in addition to obtaining a relationship for the diameters of mother and daughter arteries in a bifurcation, based on the same minimum energy hypothesis and the principle of virtual work derived another relationship for the angle between two daughter arteries. The purpose of this part is to add the energy of the oscillatory part to the calculations and extend the relationship Murray derived. The equation from which we must begin is Eq. 13 because the additional term we should add to Murray's calculation is the same we added for deriving the relationship of diameters. To avoid repetition, the final answer of the calculations will suffice but we explain the method Murray used to derive these relations. After deriving from Eq. (13) and putting Q_s^2 in Eq. (13) we get to $E_{tot}/l \sim AR^2$. Consider a bifurcation like what is shown in Fig. 5. Assume that the thick lines are the optimum case for both diameters and angles. According to Fig. 5 if a virtual movement in the junction point occurs along the mother artery (dashed lines) then it increases energy in the mother artery and decreases energy in the daughter arteries as much as $dl_0R_0^2A_0$, $dl_0R_1^2\cos(x)A_1$ and $dl_0R_2^2\cos(y)A_2$ respectively. According to the principle of virtual work because the movement is virtual so the virtual work must be zero which means $dl_0R_0^2A_0 - dl_0R_1^2\cos(x)A_1 - dl_0R_2^2\cos(y)A_2 = 0$ here. A_1 and A_2 are functions of radius and Womersley number. By another similar virtual movement along daughter arteries, we will get to two other equations. And by solving three of them at the same time, the three unknowns, cos(x), cos(y) and cos(x + y) and will be:

$$\cos(x) = \frac{R_0^4A_0^2 + R_1^4A_1^2 - R_2^4A_2^2}{2R_0^2A_0R_1^2A_1} \tag{21}$$

$$\cos(y) = \frac{R_0^4A_0^2 + R_2^4A_2^2 - R_1^4A_1^2}{2R_0^2A_0R_2^2A_2} \tag{22}$$

$$\cos(x+y) = \frac{R_0^4A_0^2 - R_1^4A_1^2 - R_2^4A_2^2}{2R_1^2A_1R_2^2A_2} \tag{23}$$

In the above equations, x and y are the angles between the along mother artery and daughter arteries. Also, $x + y$ is the angle between two daughter arteries. A_0, A_1 and A_2 are functions of Womersley number and radii of arteries. Here, like what has been calculated for diameters the effect of Womersley number and frequency as coefficients are visible clearly. Like the previous part, if we want to better examine Eq. (23), it needs to be simplified a bit and finally, it would be in the following dimensionless in Eq. (24) the terms $\frac{A_1}{A_0}$ and $\frac{A_2}{A_0}$ are similar functions of Womersley number. So with a similar argument to what was used for diameters, here, these terms are plotted for different ratios in a certain range of Womersley numbers. If $\lambda_i = A_i/A_0$ in which $\lambda_1 = A_1/A_0$ and $\lambda_2 = A_2/A_0$ corresponding α and β respectively, then we have the following diagrams:

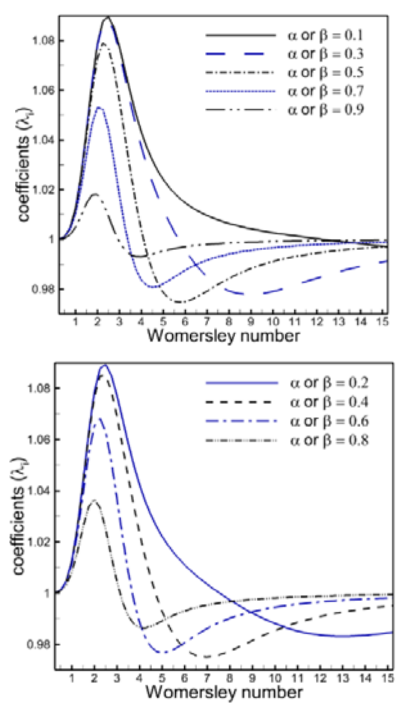

Fig. 6. Values of the coefficients λ_i according to Womersley number for different ratios α or β

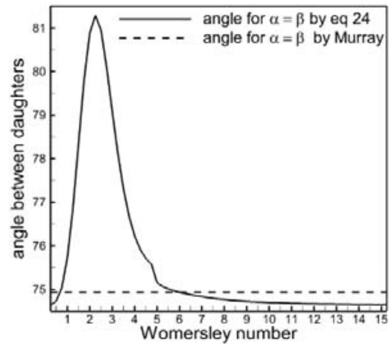

Fig. 7. Variation of angle between daughter arteries with Womersley number compared with the angle derived by Murray's law for special case which in two daughters have the same diameter

$$\cos(x+y) = \frac{(1-\alpha^4\lambda_1^2 - \beta^4\lambda_2^2)}{2\alpha^2\lambda_1\beta^2\lambda_2} \quad (24)$$

In Fig. 6 when the Womersley number tends to zero the coefficients tend to 1 just like the coefficients in Fig. 2, and the relationship (24) turns to the equation Murray derived for steady flow. The most variation occurs between Womersley numbers 2 and 5. And for larger Womersley numbers the coefficients tend to 1 again. Like what we did in the previous part for better understanding the difference, we consider a special case in which the two daughter arteries have the same radius. The following diagram is a comparison between the angle Murray calculated in this special case (almost 75 degrees) and the angle derived by Eq. (24).

Figure 7 shows the variation of angle between two daughter arteries in a symmetric bifurcation according to the Womersley number in the mother artery. According to the diagram in Fig. 7, the most variation occurs in the range 2 to 5 so that the angle increases to almost 82 degrees. But this angle is 75 degrees by Murray's law. For small Womersley numbers, the angle tends to 75 degrees according to Fig. 7, and for Womersley numbers, larger than 5 too and the reasons are the same explained before.

4. CONCLUSION

As explained in detail earlier, arterial bifurcations are very sensitive points in the cardiovascular. That is why the examination of the arterial system, especially branches and diseases related to the arterial junctions and bifurcations has been considered for a long time. But most researches are based on the assumption that the flow is steady. However, in some places, the arterial system assumes a steady flow with a very good approximation such as capillaries, but we also have to notice how the heart works. Although the average normal blood pressure of an adult varies between 80 mmHg and 120 mmHg, in the present research, the amplitude of the oscillatory part of the pressure gradient is equal to the value of the steady pressure gradient to make its role clearer. The largest variation in coefficients of Murray's generalized relationship for diameters is in the range of Womersley numbers 2 to 5, so that is a special case in which the two daughter arteries have the same diameter, the power of diameters varies up to 3.2. Similarly, for the angles, in the range of Womersley numbers between 2 and 5, there is also a difference of about 6 degrees with Murray's relation for angles. Both the relation of diameters and the relations of angles, if the Womersley number tends to zero, the relations tend to Murray's relations. The physical reason is that when the frequency drops to zero, the flow becomes a steady flow. On the other hand, as the Womersley number increase, the obtained relations get closer to Murray's relations. The physical reason is that the amplitude of the oscillating part of the velocity becomes smaller and therefore its effect will be less. It seems that the influence of Womersley number is not as much as it would be expected. But still, it can be a good and useful result. As it has been shown, dealing with equations of pulsating flow and the solution of it, is much harder than dealing with equations of steady flow. So however in some arteries flow is pulsating, but we can use the same Murray's law assuming steady flow with acceptable accuracy, at least for diameters. Notice that the result obtained in the present paper, is based on five times bigger the amplitude of the oscillatory part of velocity and flow relative to the reality, so it is reasonable that we say that Womersley number has almost no effect on Murray's law.

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