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Study of Heat and Mass Transport in Bénard-Darcy Convection with G-Jitter and Variable Viscosity Liquids in a Porous Layer with Internal Heat Source

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ABSTRACT

In this research article, we investigated the weakly non-linear effect of gravity modulation for the temperature dependent viscous fluid in a horizontal porous layer in the presence of internal heat source. We use power series expansion in terms of the amplitude of gravity modulation, which is considered to be small for double-diffusive convection in porous media. We graphically show the effect of internal heat source, solute Rayleigh number, Lewis number, Vadász number, thermo-rheological parameter, the amplitude of gravity modulation, the frequency of modulation on the heat and mass transfer using Ginzburg-Landau equation. The effect of gravity modulation is found significant and is more effective for the low values of frequency of modulation.

Keywords: Ginzburg-Landau equation; Gravity modulation; Porous media; Temperature dependent viscosity; Double-diffusive convection; Weakly non-linear stability.

NOMENCLATURE

Latin Symbols

- A amplitude of convection
- a_1 amplitude of gravity modulation
- d depth of the fluid layer
- $\mathbf{g}$ acceleration due to gravity
- $\mathbf{K}$ permeability of the porous medium
- k_c critical wave number
- Le Lewis number, $Le = \dfrac{\kappa_T}{\kappa_S}$
- Nu Nusselt number
- p reduced pressure
- Ra_T thermal Rayleigh number, $Ra_T = \dfrac{\beta_T g_0 \Delta T d \mathrm{K}}{\nu \kappa_T}$
- Ra_S solute Rayleigh number, $Ra_S = \dfrac{\beta_S g_0 \Delta S d \mathrm{K}}{\nu \kappa_T}$
- R_i internal heat source parameter $R_i = \dfrac{Qd^2}{\kappa_T}$
- S solute concentration
- ΔS solute difference across the porous layer
- Sh Sherwood number
- T temperature
- ΔT temperature difference across the porous layer
- t time

- β_T coefficient of thermal expansion
- γ heat capacity ratio $\gamma = \dfrac{(\rho c)m}{(\rho c)f}$
- δ_0 small parameter variation of viscosity with temperature
- ε perturbation parameter
- κ_S solutal diffusivity
- κ_T thermal diffusivity
- μ dynamic viscosity of the fluid
- ν kinematic viscosity, $\left(\dfrac{\mu}{\rho_0}\right)$
- ρ fluid density
- τ slow time $\tau = \varepsilon^2 t$
- φ porosity
- ψ stream function
- Ω frequency of modulation

Other symbols

- ∇_1^2 $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

V thermorheolgical parameter $V = \delta_0 \Delta T$

Va Vadász number $Va = \dfrac{\nu d^2}{K\kappa_T}$

(x,z) horizontal and vertical co-ordinates

Greek Symbols

β_S coefficient of solute expansion

∇^2 $\nabla_1^2 + \dfrac{\partial^2}{\partial z^2}$

Subscripts

b basic state

c critical

0 reference value

Superscripts

$'$ perturbed quantity

$*$ dimensionless quantity

St stationary

1. INTRODUCTION

Horton-Rogers-Lapwood convection attracts many researchers in recent years as it come across many natural convection and in most geological, physical, mechanical engineering, chemical engineering, geophysical problems. Documented review in this area are mainly provided by Ingham and Pop (2005), Vadász (2008), Nield and Bejan (2013), Vafai (2005). Double-diffusive convection in which two components having different rates of diffusivity assent to the instability arises in oceans, where heat and salt concentration exist with different gradients and diffuse at differing rates, in our atmosphere where temperature and concentrations have different diffusive rates at different altitudes with respect to the sea level, magma chambers, polymeric liquids, geothermal energy extraction, oil re-covery process, nuclear waste disposal and the migration of moisture through air contained in fibrous insulation. Kuznetsov and Nield (2008), Kuznetsov and Nield (2010), Kuznetsov and Nield (2011), Bhadauria (2012) undertook the thermal instability of double diffusive convection in porous media.

Under the micro-gravity environment gravity field is a randomly fluctuating field and thus the gravity modulation of the system leads to the variable coefficients in the governing equations of thermal instability in porous media and involves the vertical time-periodic vibrations of the system. This leads to the appearance of a modified gravity, collinear with actual gravity, in the form of a time-periodic gravity field perturbation and is known as gravity modulation or g-jitter in literature. Gravity modulation can be taken as an effective mechanism to control the heat transfer by an external regulation. Documented article in this area are given by Malashetty and Padmavathi (1997), Rees and Pop (2000), Rees and Pop (2001), Rees and Pop (2003), Govender (2005), Kuznetsov (2005), Kuznetsov (2006b), Kuznetsov (2006a), Strong (2008), Strong (2009), Razi *et al.* (2009), Vanishree and Siddheshwar (2010), Siddheshwar *et al.* (2012a), Swamy *et al.* (2013), Swamy (2014).

There are a large number of practical situations in which convection is driven by the internal heat source. Due to internal heating of earth, there is atemperature gradient between the interior and the exterior of the earth's crust which causes convection in earth crust also application of internal heat source may be found in radioactive decay of un-stable isotopes, metal waste form development for spent nuclear fuel, weak exothermic reaction which can take place within porous materials moreover internal heat source is the main energy source of celestial bodies which is generated by radioactive decay and nuclear reaction. Related research article in this area is provided by Tveitereid (1977), Bejan (1978) , Alex *et al*. (2001), Saravanan (2009), Cookey *et al*. (2010), Nouri-Borujerdi *et al*. (2007), Nouri-Borujerdi *et al*. (2008), Capone *et al*. (2011), Bhadauria (2012), Bhadauria *et al*. (2013).

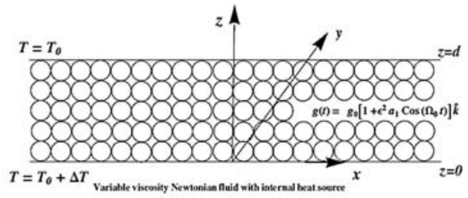
The viscosity of a fluid is one of the property which varies as temperature varies but most of the articles kept it as a constant with respect to temperature, for some fluids it is less and for some fluids it is significant. Temperature dependent viscosity fluid gives rise to variation in the top and bottom structures and referred as a non-Boussinesq effect. Nonlinear energy stability theory has been derived by Richardson and Straughan (1993) for the problem of convection in porous medium when the viscosity depends on the temperature for vanishingly small initial data thresholds, Payne *et al*. (1999) has been studied the unconditional nonlinear stability for temperature sensitive fluid in porous media, further more Payne and Straughan (2000) extend their analysis for the cases when the viscosity variation may be quadratic or when convection is penetrative. Qin and Chadam (1996) studied the nonlinear energy stability by considering the temperature dependent viscosity and inertial drag by taking higher-order approximations for the viscosity-temperature and density-temperature relation, Nield (1996), Holzbecher (1998) studied the thermal instability for the variable viscosity fluid in porous layer using the FAST-C(SD) code for numerical modeling, Rees *et al.* (2002), Siddheshwar and Chan (2004), Vanishree and Siddheshwar (2010) investigated the linear thermal instability for the temperature dependent fluid using weighted residual technique, Siddheshwar *et al.* (2012b) studied the effects of variable viscosity and the gravity modulation on the heat transfer in an anisotropic porous medium, Srivastava *et al*. (2013) analyse the non-linear effect of internal heat source and gravity modulation on the heat transfer for variable viscosity fluid in an anisotropic porous layer.

The simultaneous appearance of solid and liquid phases during the solidification of metallic alloys referred as mushy zones (Worster (1991)), also during the solidification of metals there is a temperature difference between upper and lower layer of molten metals which solidify slowly and in which the heat consumed by the molten metals works as the internal heat source. In the present paper the study of extraction of metals from ores where a mushy layer is formed during solidification of a metallic alloy is of particular interest. The quality and structure of the resulting solid during the solidification of binary alloys can be controlled by influencing the transport process externally, which can be done by thermal modulation, gravity modulation, and rotation or by internal heating. However, in the present study, internal heating and gravity modulation of the system was used as an external means to influence the transport process, thereby controlling the quality and structure of the resulting solid. If one were to quantify heat and mass transports in porous media in the presence of gravity modulations, then the linear stability analysis is inapplicable and the nonlinear stability analysis becomes inevitable. Numerical analysis of the Ginzburg-Landau equation is simpler than numerical solution of the equations of motion. In addition, analysis of stability of some simple (for example, periodic) solutions of the Ginzburg-Landau equation allows researchers to simplify the analysis of spatiotemporal dynamics of complex flows in fluid mechanics, while investigating a weakly non-linear stability of systems Ginzburg-Landau equation arise as a consequence of the solvability condition in a large category of problems in contin-uum mechanics. In the light of the above, we make a weakly nonlinear analysis of the problem using the Ginzburg-Landau equation and, in the process, quantify the heat and mass transports in terms of the amplitude governed by the Ginzburg-Landau equation. There are no reported studies on this aspect of the problem.

2. GOVERNING EQUATIONS

We consider an infinitely extended horizontal porous layer saturated by variable viscosity Newtonian fluid with temperature-dependent viscosity confined between the planes $z = 0$ and $z = d$, which is heated and salted from below. We choose Cartesian frame of reference as, origin in the lower boundary and the z-axis vertically upward direction.The gravity force is acting in vertically down-ward direction, we consider only free-free boundaries. A uniform adverse temperature gradient ΔT and concentration gradient ΔS is maintained be-tween the surfaces. Further the density variation is considered under Boussinesq approximation. The governing equations under above considerations are given by

$$\nabla.\mathrm{q} = 0, \tag{1}$$



Schematic of the flow configuration

$$\frac{\rho_0}{\varphi}\frac{\partial \mathrm{q}}{\partial t} = -\nabla p + \rho \mathrm{g}(t) - \mu \mathrm{K.q}, \tag{2}$$

$$\gamma\frac{\partial T}{\partial t} + (\mathrm{q}.\nabla)T = \kappa_T \nabla^2 T + Q(T - T_0), \tag{3}$$

$$\varphi\frac{\partial S}{\partial t} + (\mathrm{q}.\nabla)S = \kappa_S \nabla^2 S, \tag{4}$$

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0)\right], \tag{5}$$

$$\mathrm{g}(t) = g_0\left[1 + \varepsilon^2 a_1 \cos(\Omega_0 t)\right]\hat{k}, \tag{6}$$

$$\mu(T) = \frac{\mu_0}{1 + \varepsilon^2 \delta_0 (T - T_0)}, \tag{7}$$

where, **q** = (u,v,w) is velocity (m/s), φ is porosity of the matrix, p is the pressure (Pa), **g** is the acceleration due to gravity (m/s^2), μ is viscosity (Ns/m^2) and is taken as in the sense that $1/\mu$ is linear in the temperature because the steady state temperature profile in the Horton-Rogers-Lapwood problem in the vertical coordinate z is linear, this gives a situation where $1/\mu$ is linear in z, the ordinary differential equation that now arises is one whose coefficients are linear in z and this allow us to solve the equation using a comparatively simple series method. ρ is density (kg/m^3), T and S represents temperature (K) and concentration (kg/m^3) respectively. **K** is the inverse of the permeability tensor (m^2) , κ_T is the thermal diffusivity tensor (m^2/s), $Q = Q_0/(\rho c)f$ and Q_0 is the volumetric internal heat source (W/m^3) and the magnitude of the internal heat generation can vary with the temperature of the elements, ρ_0 is reference density, g_0 is mean gravity, a_1 is amplitude of gravity modulation, Ω_0 is the frequency (s^{-1}), ε is the quantity that indicates smallness in order of magnitude of modulation and t is time . Furthermore κ_S is the solute diffusivity (m^2/s), β_T is thermal expansion coefficient (K^{-1}) and β_S is density coefficient for salinity (kg^{-1}m^3), $\gamma = \frac{(\rho c)m}{(\rho c)f}$ is the heat capacity ratio where c is the specific heat (J/kg K) and subscript f and m stands for fluid and medium, also for simplicity γ and φ is taken unity in this paper. We consider only two-dimensional disturbances in our study, introducing the stream function ψ and eliminating the pressure term and then nondimensionalizing the resultant equations using the substitution

$$(x,y,z)=(x^*,y^*,z^*)d, t=t^*(\gamma d^2/\kappa_T), \quad \psi=(\kappa_T)\psi^*, T-T_0=(\Delta T)T^*, S=(\Delta S)S^* \tag{8}$$

The nondimensionlized (after dropping the asterisks for simplicity) Eqs. (2-4) are:

$$\begin{aligned}\frac{1}{Va}\frac{\partial(\nabla^2\psi)}{\partial t} &= -Ra_T(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0 t))\frac{\partial T}{\partial x} \\ &+Ra_S(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0 t))\frac{\partial S}{\partial x} \\ &-\bar{\mu}(T)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi \\ &-\left(\frac{\partial\bar{\mu}}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\bar{\mu}}{\partial z}\frac{\partial\psi}{\partial z}\right),\end{aligned} \tag{9}$$

$$\frac{\partial T}{\partial t}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+R_i\right)T=\frac{\partial(\psi,T)}{\partial(x,z)}, \tag{10}$$

$$\frac{\partial S}{\partial t}-\frac{1}{Le}\nabla^2 S=\frac{\partial(\psi,S)}{\partial(x,z)}, \tag{11}$$

where $\bar{\mu}(T)=\dfrac{1}{1+\varepsilon^2 V(T-T_0)}$ and the appearance of ε^2 indicates that the viscosity variation is weak as ε is small quantity. $Va=\dfrac{\nu d^2}{K\kappa_T}$ is the Vadász number or Darcy-Prandtl number $Ra_T=\dfrac{\beta_T g\Delta T d\mathrm{K}}{\nu\kappa_T}$ is thermal Rayleigh number, $Ra_S=\dfrac{\beta_S g\Delta S d\mathrm{K}}{\nu\kappa_T}$ is solute Rayleigh number, $R_i=\dfrac{Qd^2}{\kappa_T}$ Internal heat source parameter, $Le=\dfrac{\kappa_T}{\kappa_S}$ is Lewis number, $V=\delta_0\Delta T$ thermo-rheological parameter.

The boundary conditions for solving Eqs. (9-11) are

$$\psi=0,\ T=1 \ \ and \ \ S=1 \ \ at \ \ z=0 \tag{12}$$

$$\psi=0,\ T=0 \ \ and \ \ S=0 \ \ at \ \ z=1. \tag{13}$$

The boundaries of a porous medium can be either free or rigid. We know that under laboratory conditions free boundaries are less accessible to the experiments therefore; one has to consider the rigid boundaries. However, in real situations like geothermal regions the porous layer under consideration cannot be isolated from the surrounding region to avoid the penetration of the fluid; therefore we have to consider only free surfaces. Further, the mathematical analysis of the problem becomes easier to handle, therefore for mathematical convenience also the free-free boundary conditions are used.

The conduction profile is given by

$$\psi_b=0,\ T_b(z)=\frac{\mathrm{Sin}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}} \quad and \quad S_b(z)=1-z \tag{14}$$

Using $\bar{\mu}=\bar{\mu}(T_b)$ Eq. (9) reduces to

$$\begin{aligned}\frac{1}{Va}\frac{\partial(\nabla^2\psi)}{\partial t} &= -Ra_T(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0 t))\frac{\partial T}{\partial x} \\ &+Ra_S(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0 t))\frac{\partial S}{\partial x} \\ &-\bar{\mu}(T)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi \\ &-\frac{\partial\bar{\mu}}{\partial z}\frac{\partial\psi}{\partial z},\end{aligned} \tag{15}$$

where $\bar{\mu}(T_b)=\dfrac{\mu'}{\mu_0}=\dfrac{1}{\left[1+\varepsilon^2 V\dfrac{\mathrm{Sin}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}}\right]}$

and $\Omega_0^*=\dfrac{\Omega_0 d^2}{\kappa_{Tz}}$.

Now imposing finite amplitude perturbations on the basic quiescent state given by Eq. (14) as

$$\psi=\Psi,\ T=\frac{\mathrm{Sin}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}}+\Theta \tag{16}$$

and $S=1-z+\phi$

Substituting the above Eq. (16) in Eqs. (10, 11, 15) we have

$$\begin{aligned}\frac{1}{Va}\frac{\partial(\nabla^2\psi)}{\partial t} &= -Ra_T(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0^* t))\frac{\partial\Theta}{\partial x} \\ &+Ra_S(1+\varepsilon^2 a_1 \mathrm{Cos}(\Omega_0^* t))\frac{\partial\phi}{\partial x} \\ &-\frac{1}{\left[1+\varepsilon^2 V\dfrac{\mathrm{Sin}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}}\right]}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi \\ &-\frac{\varepsilon^2 V\sqrt{R_i}\,\mathrm{Cos}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}\left[1+\varepsilon^2 V\dfrac{\mathrm{Sin}\sqrt{R_i}(1-z)}{\mathrm{Sin}\sqrt{R_i}}\right]^2}\frac{\partial\psi}{\partial z},\end{aligned} \tag{17}$$

$$-\frac{\mathrm{dT}_b}{\mathrm{dz}}\frac{\partial\psi}{\partial x}+\frac{\partial\Theta}{\partial t}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+R_i\right)\Theta=\frac{\partial(\psi,\Theta)}{\partial(x,z)}, \tag{18}$$

$$\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial t}-\frac{1}{Le}\nabla^2\phi=\frac{\partial(\psi,\Theta)}{\partial(x,z)} \tag{19}$$

Boundary conditions to solve Eqs. (17-19) are

$$\psi = 0,\ \Theta = 0 \text{ and } \phi = 0 \ \ at \ \ z = 0, \tag{20}$$

$$\psi = 0,\ \Theta = 0 \text{ and } \phi = 0 \ \ at \ \ z = 1. \tag{21}$$

We now introduce the following asymptotic expansion

$$Ra_T = R_{0c} + \varepsilon^2 R_2 + \varepsilon^4 R_4 + ..., \tag{22}$$

$$\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + ..., \tag{23}$$

$$\Theta = \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + ..., \tag{24}$$

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + ..., \tag{25}$$

where R_{0c} is the critical value of the Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

Since the exponential growth of linearised disturbances is proportional to the difference of the Rayleigh number from the critical value therefore, we assume the variation of time only at the slow time scale $\tau = \varepsilon^2 t$, and arranging the systems at different order of ε.

At the lowest order, we have

$$\begin{pmatrix} \nabla_1^2 & R_{0c}\dfrac{\partial}{\partial x} & -Ra_S\dfrac{\partial}{\partial x} \\ -\dfrac{dT_b}{dz}\dfrac{\partial}{\partial x} & -\left(\nabla_1^2 + R_i\right) & 0 \\ \dfrac{\partial}{\partial x} & 0 & -\dfrac{1}{Le}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \Theta_1 \\ \phi_1 \end{pmatrix} = 0, \tag{26}$$

Solution at the lowest order subject to the boundary conditions (20, 21) are given by

$$\psi_1 = A[\tau]\mathrm{Sin}\left(k_c x\right)\mathrm{Sin}(\pi z), \tag{27}$$

$$\Theta_1 = -\frac{4\pi^2 k_c}{\left(\delta^2 - R_i\right)\left(R_i - 4\pi^2\right)} A[\tau]\mathrm{Cos}\left(k_c x\right)\mathrm{Sin}(\pi z), \tag{28}$$

$$\phi_1 = \frac{-k_c Le}{\delta^2} A[\tau]\mathrm{Cos}\left(k_c x\right)\mathrm{Sin}(\pi z), \tag{29}$$

where $\delta^2 = k_c^2 + \pi^2$ is the critical wave number.

The critical value of the Rayleigh number and the corresponding wave number for the onset of stationary convection is calculated numerically and the expression for Rayleigh number is given by

$$R_{0c} = \frac{\left(\delta^2 - R_i\right)\left(4\pi^2 - R_i\right)\left(\delta^4 + k_c^2 Ra_S Le\right)}{4\pi^2 k_c^2 \delta^2} \tag{30}$$

2.1 Amplitude Equation And Heat And Mass Transport For Stationary Instability

At the second order, we have

$$\begin{pmatrix} \nabla_1^2 & R_{0c}\dfrac{\partial}{\partial x} & -Ra_S\dfrac{\partial}{\partial x} \\ -\dfrac{dT_b}{dz}\dfrac{\partial}{\partial x} & -\left(\nabla_1^2 + R_i\right) & 0 \\ \dfrac{\partial}{\partial x} & 0 & -\dfrac{1}{Le}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \Theta_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} R_{21} \\ R_{22} \\ R_{23} \end{pmatrix}, \tag{31}$$

where

$$R_{21} = 0, \tag{32}$$

$$R_{22} = \frac{2\pi^3 k_c^2}{\left(\delta^2 - R_i\right)\left(R_i - 4\pi^2\right)} A[\tau]^2 \,\mathrm{Sin}\left(2\pi z\right), \tag{33}$$

$$R_{23} = \frac{-k_c^2 Le\pi}{2\delta^2} A[\tau]^2 \,\mathrm{Sin}\left(2\pi z\right) \tag{34}$$

The second order solution subject to the boundary condition (20, 21), is given by

$$\psi_2 = 0, \tag{35}$$

$$\Theta_2 = -\frac{2\pi^3 k_c^2}{\left(\delta^2 - R_i\right)\left(4\pi^2 - R_i\right)^2} A[\tau]^2 \,\mathrm{Sin}\left(2\pi z\right), \tag{36}$$

$$\phi_2 = -\frac{k_c^2 Le^2}{8\pi\delta^2} A[\tau]^2 \,\mathrm{Sin}\left(2\pi z\right), \tag{37}$$

The horizontally averaged Nusselt number and Sherwood number, Nu and Sh, for stationary mode of convection (the mode considered in this problem) is given by :

$$Nu(\tau) = 1 + \frac{\left[\dfrac{k_c}{2\pi}\displaystyle\int_0^{\frac{2\pi}{k_c}}\left(\frac{\partial\Theta_2}{\partial z}\right)dx\right]_{z=0}}{\left[\dfrac{k_c}{2\pi}\displaystyle\int_0^{\frac{2\pi}{k_c}}\left(\frac{dT_b}{dz}\right)dx\right]_{z=0}}, \tag{38}$$

$$Sh(\tau) = 1 + \frac{\left[\dfrac{k_c}{2\pi}\displaystyle\int_0^{\frac{2\pi}{k_c}}\left(\frac{\partial\phi_2}{\partial z}\right)dx\right]_{z=0}}{\left[\dfrac{k_c}{2\pi}\displaystyle\int_0^{\frac{2\pi}{k_c}}\left(\frac{dS_b}{dz}\right)dx\right]_{z=0}}, \tag{39}$$

One can notice here that the gravity modulation is effective at $O(\varepsilon^2)$ and effects $Nu(\tau)$ and $Sh(\tau)$ through $A[\tau]$ as shown next. Substituting expressions of Θ_2 and ϕ_2 in the above expression (38, 39) and simplifying, we get

$$Nu(\tau)=1+\frac{\left(4\pi^4k_c^2\,\text{Sin}\sqrt{R_i}\right)}{\left(\sqrt{R_i}\,\text{Cos}\sqrt{R_i}\left(\delta^2-R_i\right)\left(R_i-4\pi^2\right)^2\right)}\left(A[\tau]\right)^2, \quad (40)$$

$$Sh(\tau)=1+\frac{Le^2k_c^2}{4\delta^2}\left(A[\tau]\right)^2 \quad (41)$$

At the third order, we have

$$\begin{pmatrix} \nabla_1^2 & R_{0c}\frac{\partial}{\partial x} & -Ra_S\frac{\partial}{\partial x} \\ -\frac{dT_b}{dz}\frac{\partial}{\partial x} & -\left(\nabla_1^2+R_i\right) & 0 \\ \frac{\partial}{\partial x} & 0 & -\frac{1}{Le}\nabla^2 \end{pmatrix} \begin{pmatrix}\psi_3\\ \Theta_3\\ \phi_3\end{pmatrix}=\begin{pmatrix}R_{31}\\ R_{32}\\ R_{33}\end{pmatrix}, \quad (42)$$

where

$$\begin{aligned} R_{31}=&-\frac{1}{Va}\frac{\partial}{\partial\tau}\left(\nabla^2\psi_1\right) \\ &-V\frac{\text{Sin}\sqrt{R_i}\,(1-z)}{\text{Sin}\sqrt{R_i}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi_1 \\ &-V\frac{\text{Cos}\sqrt{R_i}\,(1-z)}{\text{Sin}\sqrt{R_i}}\frac{\partial\psi_1}{\partial z} \\ &-\left(R_{0c}\left(2V\frac{\text{Sin}\sqrt{R_i}\,(1-z)}{\text{Sin}\sqrt{R_i}}+a_1\,\text{Cos}(\Omega\tau)\right)\right. \\ &\left.+R_2\right)\frac{\partial\Theta_1}{\partial x} \\ &+Ra_S\left(2V\frac{\text{Sin}\sqrt{R_i}\,(1-z)}{\text{Sin}\sqrt{R_i}}+a_1\,\text{Cos}(\Omega\tau)\right) \\ &\frac{\partial\phi_1}{\partial x}, \end{aligned} \quad (43)$$

$$R_{32}=\frac{\partial\psi_1}{\partial x}\frac{\partial\Theta_2}{\partial z}-\frac{\partial\Theta_1}{\partial\tau}, \quad (44)$$

$$R_{33}=\frac{\partial\psi_1}{\partial x}\frac{\partial\phi_2}{\partial z}-\frac{\partial\phi_1}{\partial\tau} \quad (45)$$

and $\Omega=\frac{\Omega_0^*}{\varepsilon^2}=\frac{\Omega_0 d^2}{\varepsilon^2\kappa_{Tz}}$.

Substituting the value of $\psi_1,\Theta_1,\Theta_2,\phi_1$ and ϕ_2 in the above equations to get the expressions of R_{31},R_{32},R_{33}.

Applying the solvability condition for the existence of third order solution,

$$\int_0^1\int_0^{\frac{2\pi}{k_c}}\left(R_{31}\hat{\psi}_1+R_{32}\hat{\Theta}_1+R_{33}\hat{\phi}_1\right)dxdz=0 \quad (46)$$

where (∧) represents adjoint solution of the first-order system, we get the non-autonomous Ginzburg-Landau equation with time periodic coefficients in the form

$$A_1A'[\tau]+A_2A[\tau]+A_3\left(A[\tau]\right)^3=0, \quad (47)$$

where

$$A_1=\left(\frac{\delta^2}{Va}+\frac{\left(\delta^4+k_c^2LeRa_S\right)}{\delta^2\left(\delta^2-R_i\right)}-\frac{k_c^2Le^2Ra_S}{\delta^4}\right), \quad (48)$$

$$\begin{aligned} A_2=&-\left(1+a_1\,\text{Cos}[\Omega\tau]\right)\frac{\left(\delta^4+k_c^2LeRa_S\right)}{\delta^2} \\ &+a_1\,\text{Cos}[\Omega\tau]\frac{k_c^2LeRa_S}{\delta^2} \\ &-\frac{\delta^2V}{\text{Sin}\left[\sqrt{R_i}\right]}\left(\frac{4\pi^2\left(-1+\text{Cos}\left[\sqrt{R_i}\right]\right)}{\sqrt{R_i}\left(R_i-4\pi^2\right)}\right. \\ &\left.-\frac{2\pi^2V\sqrt{R_i}\left(-1+\text{Cos}\left[\sqrt{R_i}\right]\right)}{\left(4\pi^2-R_i\right)\text{Sin}\left[\sqrt{R_i}\right]}\right), \end{aligned} \quad (49)$$

$$A_3=\left(\frac{\pi^2k_c^2\left(\delta^4+k_c^2LeRa_S\right)}{\delta^2\left(\delta^2-R_i\right)\left(4\pi^2-R_i\right)}-\frac{k_c^4Le^3Ra_S}{8\delta^4}\right). \quad (50)$$

The Ginzburg-Landau equation given by (47) is a Bernoulli equation and obtaining its analytical solution is difficult due to its non-autonomous nature. Therefore, it has been solved numerically by the in-built function NDSolve of Mathematica 7.0, subject to the initial condition $A[0]=a_0$, where a_0 is the chosen initial amplitude of convection. In our calculations, we may assume $R_2=R_0$ to reduce the parameters by one.

3. RESULTS AND DISCUSSIONS

We perform weakly nonlinear analysis of gravity modulation for temperature dependent viscosity fluid in closely packed porous media, there-fore, the Darcy model is considered for the governing equation for linear momentum. The work of Nield (1996) has been used for the thermo-rheological relationship of temperature dependent viscosity of the fluid. We investigated the effect gravity modulation and thermo-rheological parameter on heat and mass transport. We consider the effect of gravity modulation to be of order $O(\varepsilon^2)$ this will provide us only small amplitude vibrations. Such an assumption will help us in obtaining the amplitude equation of convection in a rather simple and elegant manner and is much easier to obtain than in the case of the Lorenz model. In order to study the heat and mass transfer, we need to perform the nonlinear analysis. We use gravity modulation as an external regulation of convection to control the heat and mass transfer as desired by the need. We consider the gravity modulation, internal heat source and temperature dependent viscous fluid for either enhancing or inhibiting convective heat transport as is required by a real application.

Vadász (1998) pointed out that there are some modern porous medium applications, such as mushy layer in solidification of binary alloys and fractured porous medium, where the value of Va may be considered to be of unity order, therefore the time-derivative term in the present study has been retained. Further this is the reason that we have kept the values of Va around one in our calculations, and retained the local acceleration term $\frac{1}{Va}\frac{\partial q}{\partial t}$.

The values of $Nu(\tau)$ and $Sh(\tau)$ are obtained numerically from the expressions of $Nu(\tau)$ and $Sh(\tau)$ by using the numerical value of amplitude obtained from the Ginzburg-Landau equation. We use the values to plot the curve for $Nu_C(\tau)$ and $Sh_C(\tau)$ versus τ and is presented in the Figs. 1–14. Figures 1-7 correspond to heat transfer and Figs. 8-14 correspond to mass transfer. A close observation of Eqs. (40, 41) in conjunction with Eq. (47) reveals that $Nu(\tau)$ and $Sh(\tau)$ depends on Lewis number Le, solute Rayleigh number Ra_S, Vadász number Va, thermo-rheological parameter and amplitude of g-jitter.

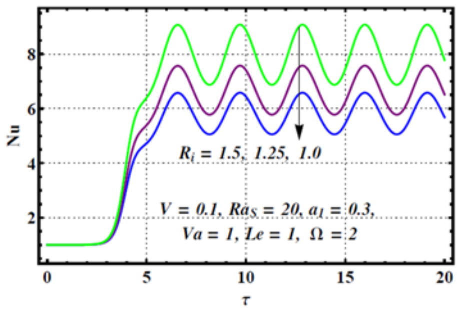


Fig. 1. Variation of Nusselt number with time for different values of R_i

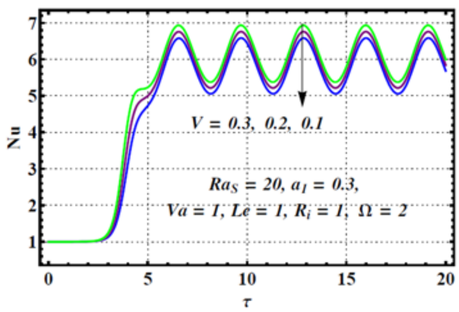


Fig. 2. Variation of Nusselt number with time for different values V

From the Figs. 1–14 it is observed that initially the value $Nu_C(\tau)$ and $Sh_C(\tau)$ is one showing that, initially there is no convection and heat transport is prevailed by conduction alone and as time increases heat and mass transport increases shows that convective regime takes place and remains oscillatory for further elapses of time. We take $\varepsilon = 0.3$ for numerical computations.

From Fig. 1 we observe that to increase in the value of internal heat source parameter the heat transfer increases it means that if we increase the strength of the internal heat source heat transfer increases. Figure 2 shows the variation of Nusselt number with time for the different values of the thermo-rheological parameter and is clear that to increase in the value

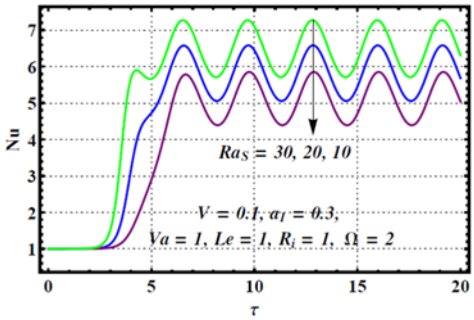


Fig. 3. Variation of Nusselt number with time for different values Ra_S

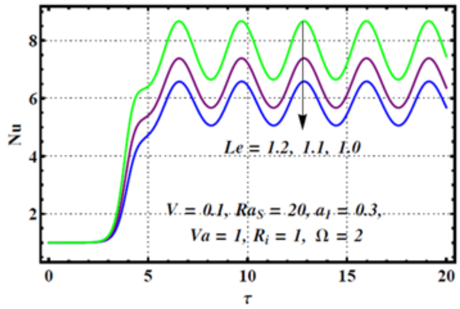


Fig. 4. Variation of Nusselt number with time for different values Le

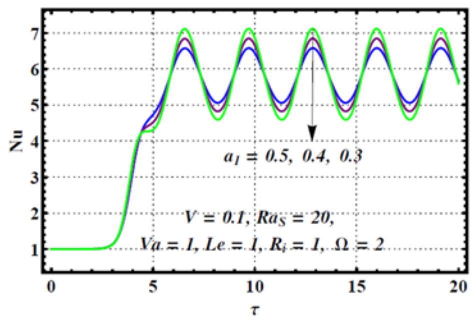


Fig. 5. Variation of Nusselt number with time for different values a_1

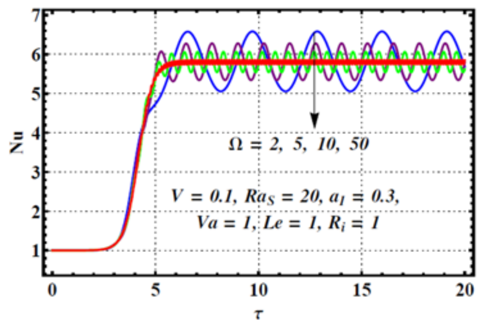


Fig. 6. Variation of Nusselt number with time for different values Ω

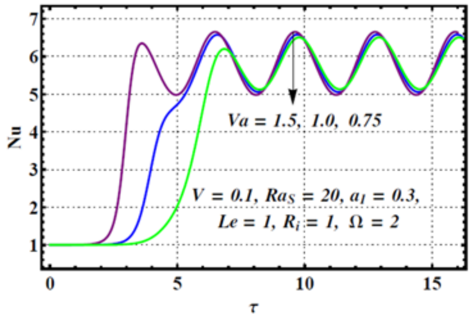


Fig. 7. Variation of Nusselt number with time for different values *Va*

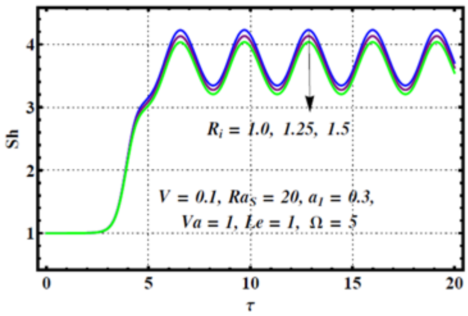


Fig. 8. Variation of Sherwood number with time for different values of R_i

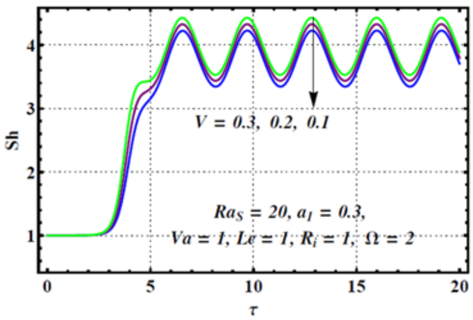


Fig. 9. Variation of Sherwood number with time for different values *V*

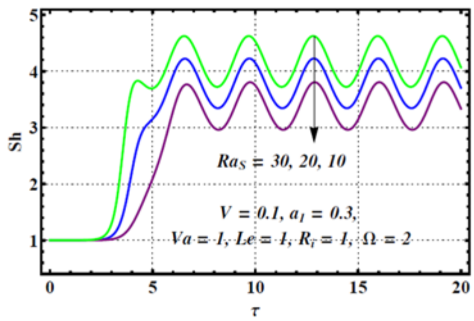


Fig. 10. Variation of Sherwood number with time for different values Ra_S

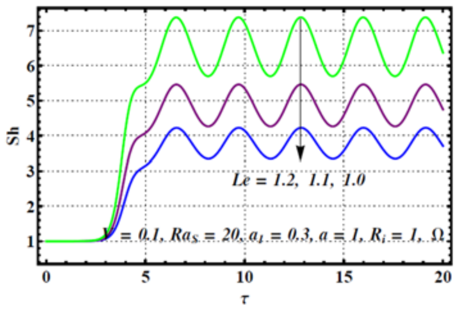


Fig. 11. Variation of Sherwood number with time for different values *Le*

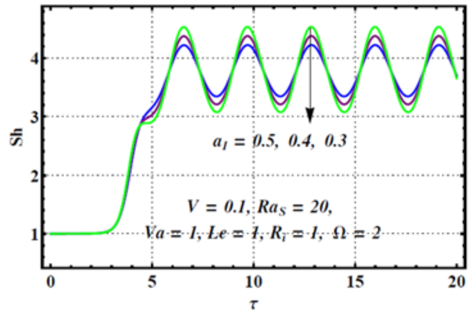


Fig. 12. Variation of Sherwood number with time for different values a_1

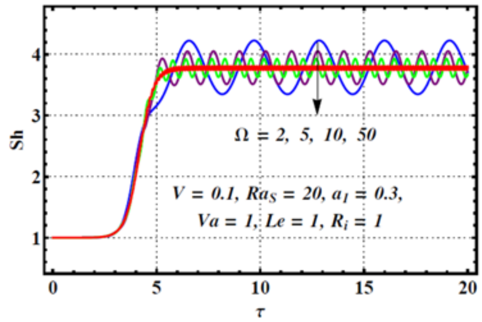


Fig. 13. Variation of Sherwood number with time for different values Ω

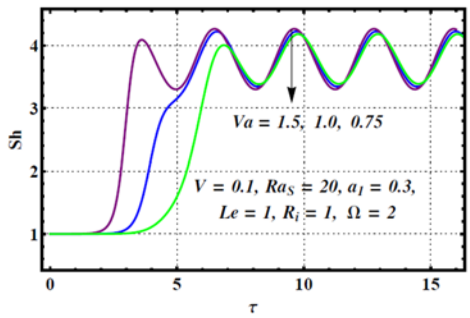


Fig. 14. Variation of Sherwood number with time for different values *Va*

of thermo-rheological parameter the heat transfer increases this indicates that fluids having high viscosity variation enhancement in the heat transfer is high. Figure 3 reveals that for the increasing value of solute Rayleigh number heat transfer increases that is, if we increase the so-lute gradient the heat transfer increases. Figure 4 shows the effect of Lewis number on the Nusselt number with respect to time and is observed that from the graph that to increase in the Lewis number the heat transfer increases it means that, if we fix the solutal diffusivity and increase the thermal diffusivity or decrease the solutal diffusivity and fix the thermal diffusivity heat transfer increases. From Fig 5 we observe that to increase in the amplitude of modulation heat transfer increases it means that if the difference between actual gravity and mean gravity increases heat transfer increases. Figure 6 shows the effect of the frequency of modulation on the heat transfer and it is observed that for the increasing value of frequency of modulation heat transfer decreases this indicates that, if we decrease the time period of the oscillation heat transfer de-creases therefore we can control the heat transfer either to enhance or to inhibit by choosing appropriate frequency of modulation as required by the real life applications, furthermore, for the higher value of the frequency of modulation the effect of gravity modulation become weak this means that for the system having small time period for gravity modulation does not produce significant variation on the heat transfer also the effect of gravity modulation disappears for the higher values of frequency of modulation. From Fig 7 shows that for the increasing value of Vadász number heat transfer in-creases that is, if we increase the Prandtl number and fix the Darcy number or fix the Prandtl number and decrease the Darcy number the heat transfer increases, furthermore, it is observed that Vadász number is more effective on the heat transfer for its lower values so we keep it around unity in our numerical computations.

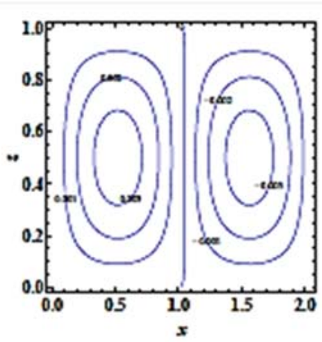
Figure 8 shows the variation of Sherwood number for the different value of internal heat source parameter with respect to time and is clear from the graph that for the increasing value of internal heat source parameter the mass transfer decreases that is if increase the strength of the heat source mass transfer decreases. From Fig. 9 we observe that to increase in thermo-rheological parameter mass transfer increases it means that for the fluids having viscosity variation high the rate of mass transfer high. From Fig. 10 it is clear that for the increasing value of solute Rayleigh number mass transfer increases this means that if we increase the solute gradient the mass transfer increases. Figure 11 reveals the effect of the Lewis number on the Sher-wood number with respect to time and it is cleat that from the graph that to increase in Lewis number mass transfer increases this indicates that if we fix the solutal diffusivity and increase the thermal diffusivity or decrease the solutal diffusivity and fix the thermal diffusivity mass transfer increases. Figure 12 shows that for the increasing value of the amplitude of modulation mass transfer increases it means that if the difference between actual gravity and mean gravity increases mass transfer in-creases. From Fig. 13 we observe that the effect of frequency of modulation is to decrease the mass transfer for its increasing value this means that if we decrease the time period of the oscillation mass transfer decreases therefore, we can choose the frequency of gravity modulation to control the mass transfer either to enhance or to inhibit as required by the real life applications, furthermore, for the higher value of frequency of modulation the effect of gravity modulation is frail on the mass transfer. From Fig. 14 it is clear that to increase in the value of Vadász number mass transfer increases it means that if we increase the Prandtl number and fix the Darcy number or fix the Prandtl number and decrease the Darcy number the mass transfer increases.

Variation of streamlines, isotherms, isohalines at different instant of time is shown graphically in Figs. 15–17. From Figs. 15a–15d it is clear that the magnitudes of streamlines initially increases as time increases and after reaching a certain value it starts oscillations, Fig. 16a–16d shows the variation of isotherms at the different instant of time and if found that from the graph initially isotherms are flat and parallel shows the heat transport is only by conduction and as time increases isotherms starts oscillating showing convective regime is in place and then forms contour showing that as time increases convection contributes in heat transport, similar behaviour is observed for isohalines in Fig. 17a–17d, moreover, it is clear that from the Figs. 15–17 after reaching some instant there is no changes in streamlines, isotherms, isohalines.

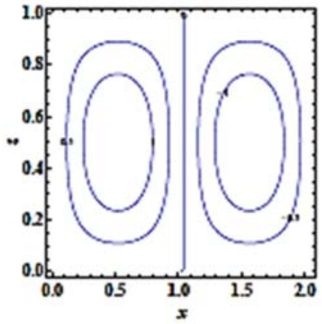
4. CONCLUSION

We perform the weakly nonlinear analysis using the Ginzburg-Landau equation for double diffusive convection in an infinite horizontal porous layer which is heated and salted from below, saturated with temperature sensitive fluid in the presence of gravity modulation and internal heat source. We studied the effect of gravity modulation and temperature dependent viscosity on the heat and mass transfer. We found that the gravity modulation is an effective mechanism to control the heat and mass transfer. The effect of frequency of modulation is significant to control the heat and mass transfer for the physical problem, furthermore for the large value of frequency of modulation the effect of gravity modulation is very frail. We found that the Vadász number is more pronounced on the heat and mass transfer when we take its value around unity, for its higher values the effect of Vadász number is not significant on the heat and mass transfer, however, the effect of gravity modulation keeps its own nature. The following conclusions has been made from our analysis, for the increasing values of parameter:

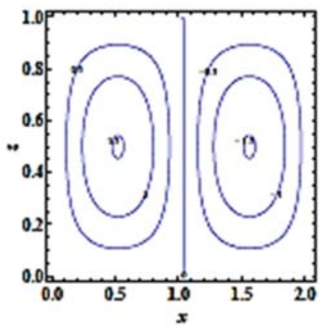
1. Internal heat source parameter R_i : heat transfer increases, mass transfer decreases.
2. Thermo-rheological parameter V : heat transfer increases, mass transfer increases.
3. Lewis number Le : heat transfer increases, mass transfer increases.
4. Solute Rayleigh number RaS : heat transfer increases, mass transfer increases.
5. Vadász number Va : heat transfer increases, mass transfer increases.



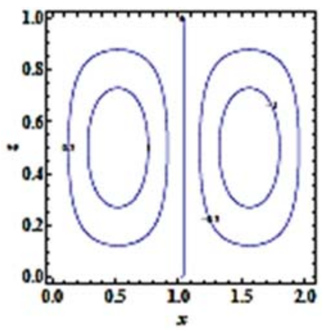
(a) τ = 0.1



(b) τ = 6

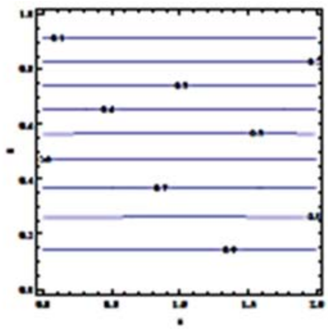


(c) τ = 7

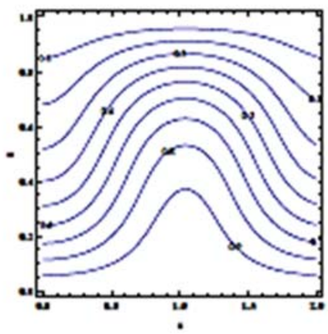


(d) τ = 8

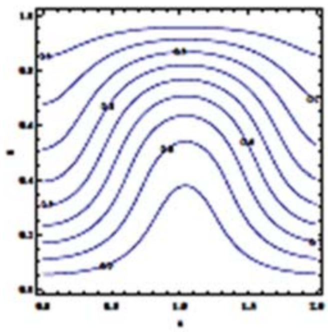
Fig. 15. Variation of stream lines with time



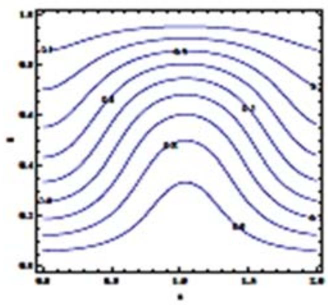
(a) τ = 0.01



(b) τ = 6



(c) τ = 7



(d) τ = 8

Fig. 16. Variation of isotherms with time

6. Frequency of modulation Ω: heat transfer decreases, mass transfer decreases.

7. amplitude of modulation a1: heat transfer increases, mass transfer increases.

ACKNOWLEDGMENTS

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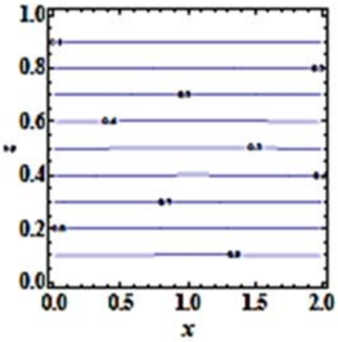
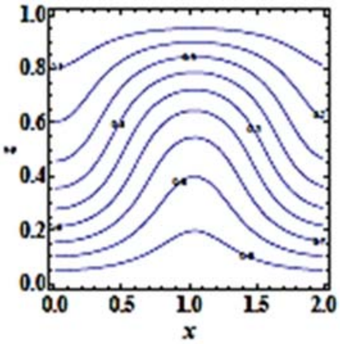


(a) τ = 0.01



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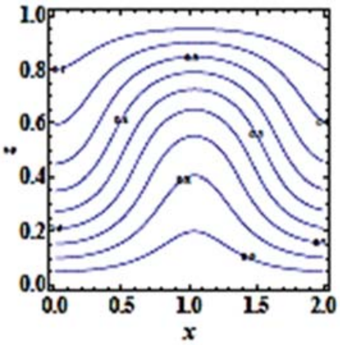
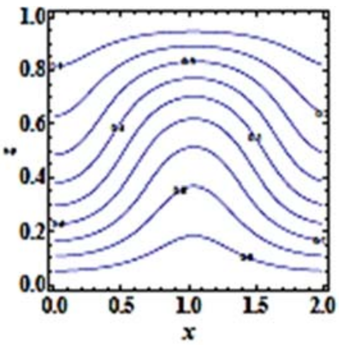


(c) τ = 7



(d) τ = 8

Fig. 17. Variation of isoconcentrations with time

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