Journal of Applied Fluid Mechanics, Vol. 11, No. 5, pp. 1433-1441, 2018.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.29252/jafm.11.05.28862

Two Semi-Analytical Methods Applied to Hydrodynamic Stability of Dean Flow

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(Received February 7, 2018; accepted April 8, 2018)

ABSTRACT

Hydrodynamic stability of Dean flow is studied using two semi-analytical methods of differential transform method (DTM) and Homotopy perturbation method (HPM). These two methods are evaluated to examine the effectiveness and accuracy of the solution of considered eigenvalue problem. Very good accordance is achieved between our semi-analytical results compared to existing numerical data. Based on our analysis, in the similar number of truncated terms, HPM is more accurate in comparison with DTM. We also concluded that for the higher wave numbers, HPM provide more accurate results with less truncated terms compared to the DTM. Finally, we found the critical Dean number 35.927 corresponding to wave number of 3.952 for onset of instability of Dean flow.

Keywords: Hydrodynamic stability; Dean flow; Differential transform method; Homotopy perturbation method.

1. INTRODUCTION

Dean flow is a common phenomenon in nature and different macro-scale industries (Bottaro 1993; Norouzi *et al.* 2014) including internal cooling in turbines blades, fluid machineries (Kumar and Nigam 2005), turbo-compressor devices, heat exchangers (Aider *et al.* 2005) and etc. Moreover, study on hydrodynamic stability of Dean flow has a chnology, T value to improve of the micro/nano electromechanical systems (MEMS/NEMS) (Fischer *et al.* 2015; Norouzi and Biglari 2013) such as lab-on-a chip devices (Kemna *et al.* 2012; Yang *et al.* 2011). In Dean flow, secondary flow as a motion of counter-rotating vortex superimposed on the stream-wise flow is produce by centrifugal forces (Drazin 2004). Imbalance between gradient of pressure and the centrifugal forces in radial direction cause to hydrodynamic instability of Dean flow under specific critical flow condition. By this instability, an extra pair of counter-rotating vortices will be observed.

In literature, an additional vortex is identified as Dean Vortices (Dean 1928). Hydrodynamic stability of Dean flow is comprehensively investigated by various researchers experimentally, numerically and mathematically (Joseph *et al.* 1975; Cheng *et al.* 1976; Ghia and Sokhey 1977; Crane and Burley 1976; Dennis 1982; Hille *et al.* 1985; Cookson *et al.* 2010; Li *et al.* 2016). Calculation of spatially evolving modes, where these modes are function of wave number (i.e. eigenvalue parameter) is the major approach to evaluate of eigenvalue problem corresponding to linear hydrodynamic stability (Drazin 2004).

A conventional discretization method such as finite difference method (FDM) is nedded to solve characteristics value problems via matrix methods, numerically. For example, hybrid theoretical–analytical approach is used to investigate of Dean flow by Zhiming and Yulu (1997). They linearized the Navier-Stokes equations via perturbation method. Then linearized equations are solved by FDM and Galerkin method. Stability of Dean flow using Galerkin method with a simple set of polynomial expansion functions is investigated by Walowit *et al.* (1964). Deka and Takhar (2004) evaluated the stability of Dean flow with Runge–Kutta approach incorporated with shooting method. Accuracy and efficiency of Chebyshev spectral methods to solve linear eigenvalue problems was investigated by Orszagl (1971). Moreover, the capability of Chebyshev spectral methods in study of Dean flow is studied by some scholars (Yamamoto *et al.* 1998; Mondal *et al.* 2015; Helal *et al.* 2016). For example, Yanase *et al.* (2002) conducted a study on the laminar Dean flow stability using of spectral method. Under the

symmetry condition, they found the steady solutions using method of Newton-Raphson. Several numerical methods to solve eigenvalue problem are presented by Canuto *et al*. (1988). They found that difficulties due to applying the boundary conditions in Chebyshev collocation matrix method, Chebyshev-tau method is more suitable. Deka and Paul (2013) studied the Dean flow via DTM, semi-analytically. As regards, they didn't evaluate the effectiveness and accuracy of their semi-analytical solution under different truncated terms of the solution.

Based on cited works and to our best of knowledge, the prior researchers did not examine the effectiveness and accuracy of different semi-analytical approaches for solving the eigenvalue problem corresponding to linear hydrodynamic stability of Dean Flow. Therefore, main objective of the current study is to evaluate of two different semi-analytical methods to investigate of Dean flow stability. To this accomplishment, the differential transformation method (DTM) by Zhou (1986) and the Homotopy perturbation method (HPM) by He (2005) as efficient methods for solving eigenvalue problems are considered. We found that the HPM solves the eigenvalue problem corresponding to linear hydrodynamic stability of Dean flow with less computational cost (i.e. lower truncated terms required to approach to the archival data) compared to DTM. As regards, higher accuracy (i.e. high order) of the HPM compared to the DTM is the main advantage of the HPM.

2. PHYSICAL MODEL OF THE PROBLEM

As illustrated in Fig. 1, we consider the steady, incompressible and viscous flow through the curved path, where $d << 0.5\left(R_2+R_1\right)$. In Fig. 1, r , θ and Z are usual cylindrical coordinates, where the Z axis coincides by the axis of the cylinder. Inner and outer radiuses are identified by R_1, R_2, respectively.

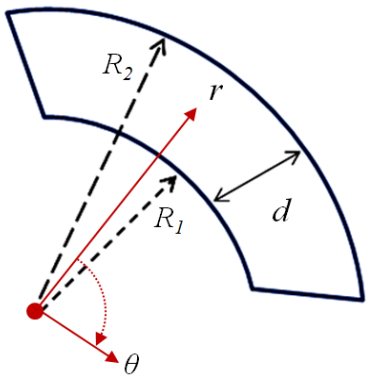

Fig. 1. Schematic of the problem.

We have steady form of Navier-Stokes equation in case of velocity vector of $U=\left(u_r,u_\theta,u_z\right)$ and fluid pressure p and under the condition of uni-directional fluid flow, as follows

$$\begin{aligned} &u_r=u_z=0;\\ &u_\theta=V(r);\\ &\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{V^2}{r};\\ &\frac{1}{\rho r}\left(\frac{\partial p}{\partial \theta}\right)_0=\upsilon DD_* V \end{aligned} \tag{1}$$

where

$$\begin{aligned} &D_*=\frac{d}{dr}+\frac{1}{r}\\ &D=\frac{d}{dr}. \end{aligned} \tag{2}$$

And ρ, υ and $\left(\frac{\partial p}{\partial \theta}\right)_0$ are density, kinematic viscosity an constant azimuthal gradient of pressure, respectively. Based on Eq. (1), velocity distribution of the basic flow is in form of parabolic plane Poiseuille flow as follows

$$V(x)=6V_m g(x) \tag{3}$$

where

$$\begin{aligned} &x=\frac{r-R_1}{d}=\frac{r-R_1}{R_2-R_1}\\ &V_m=-\frac{d}{12\rho\upsilon R_2}\left(\frac{\partial p}{\partial \theta}\right)_0\\ &g(x)=x(1-x) \end{aligned} \tag{4}$$

If we consider a spatially periodic small perturbation (v) mounted on the motion (i.e. under steady condition) of Eq. (2), we have

$$u_\theta(r,z,t)=V+v=V+e^{\sigma t}v(r)\cos\lambda z. \tag{5}$$

According to Eq. (5), the stability of fluid flow motion is significantly dependent on the exponential temporal factor σ (Walowit *et al.* 1964). In the current study, instability of Dean flow is evaluated via marginal spatially state $\sigma=0$. Linearized disturbance equations for Dean flow stability can be obtained by substitution of Eqs. (4) and (5) into Eq. (1) and some manipulations as follows (Drazin 2004; Deka and Paul 2013):

$$\begin{cases} \left(D^2-\alpha^2\right)^2 u=\alpha^2\Lambda g(x)v\\ \left(D^2-\alpha^2\right)v=(1-2x)u \end{cases} \tag{6}$$

where the boundary conditions are

$$u=Du=v=0 \quad @ \quad x=0,1 \tag{7}$$

here, $u(x)$ and $v(x)$ are velocity functions, $D=d/dx$ is an operator, $\alpha=\lambda d$ is real constant wave number, $g(x)=x(1-x)$ and $\Lambda=72Dn^2$ where Dn is the well-known Dean number (Drazin 2004; Mahapatra *et al*. 2009). Dean number defined

by $Dn = \mathrm{Re}\sqrt{d/R_1}$ where, d is the gap between the parallel curved walls, R_1 is the inner radius of the curved walls and Re is the azimuthal Reynolds number that is $\mathrm{Re} = V_m d/\upsilon$. Linear hydrodynamic stability of Dean flow Eq. (6) may be redefine by the sixth order equation as follows

$$\left(D^2-\alpha^2\right)^3 v = \alpha^2 \Lambda (x(1-x)(1-2x))v \tag{8}$$

with the boundary conditions

$$v = (D^2-\alpha^2)v = D(D^2-\alpha^2)v = 0 \;\; @ \;\; x = 0,1 \tag{9}$$

Continued, we evaluated two semi-analytical methods of DTM and HPM for finding the critical Dean number (Dn_c) at a specified value of α via solving the Eqs. (8) and (9). It should be noted that the critical Dean number ($Dn_c = \left(\Lambda_c/72\right)^{0.5}$) is corresponding to the minimum of eigenvalue parameter Λ_c (Mahapatra *et al*. 2009).

For the convenience of the readers, basic concepts of the DTM and the HPM are presented in Appendixes A and B. For more detail about our considered method, readers can refer to original paper of Zhou (1986) for the DTM and pioneering paper of He (2005) for the HPM.

3. SOLUTION OF THE PROBLEM BY DTM AND HPM

In the present section, the DTM and the HPM are used to solve Eq. (8) corresponding to the boundary conditions Eq. (9).

3.1 Solution by DTM

If we change the variables $v \to f$ and $x \to \eta$, Eqs. (8) and (9) may be redefine as

$$\begin{aligned} &f^{(6)}(\eta) - \alpha^6 f(\eta) - 3\alpha^2 f^{(4)}(\eta) + 3\alpha^4 f^{n}(\eta) \\ &-\alpha^2 \Lambda (2\eta^3 - 3\eta^2 + \eta) f(\eta) = 0 \end{aligned} \tag{10}$$

And the considered boundary conditions are

$$BC\begin{cases} f(0) = 0 \quad f''(0) - \alpha^2 f(0) = 0 \\ f'''(0) - \alpha^2 f'(0) = 0 \quad f(1) = 0 \\ f''(1) - \alpha^2 f(1) = 0 \quad f'''(1) - \alpha^2 f'(1) = 0 \end{cases} \tag{11}$$

By implementation of differential transform into Eq. (10) according to the fundamental mathematical operations of DTM (see Table A1 in Appendix A), recurrence relation is obtained as follows

$$\begin{aligned} &\frac{(k+6)!}{k!} f(k+6) - \alpha^6 f(k) - 3\alpha^2 \frac{(k+4)!}{k!} f(k+4) \\ &+3\alpha^4 \frac{(k+2)!}{k!} f(k+2) \\ &-\alpha^2 \Lambda \begin{pmatrix} 2\sum_{i=0}^{k} \delta(l-3) f(k-l) \\ -3\sum_{i=0}^{k} \delta(l-2) f(k-l) + \sum_{i=0}^{k} \delta(l-1) f(k-l) \end{pmatrix} = 0 \end{aligned} \tag{12}$$

According to DTM (Hassan 2002; Chen and Ho 1996), the boundary conditions of Eq. (11) are as follows

$$\begin{aligned} &for \quad \eta = 0; \begin{cases} f(0) = 0; \\ f(2) = 0; \\ 6f(3) - \alpha^2 f(1) = 0; \end{cases} \\ &for \quad \eta = 1; \begin{cases} \sum_{k=0}^{N} f(k) = 0; \\ \sum_{k=0}^{N} \left[k(k-1) - \alpha^2 \right] f(k) = 0 \\ \sum_{k=0}^{N} \left[k(k-1)(k-2) - \alpha^2 k \right] f(k) = 0 \end{cases} \end{aligned} \tag{13}$$

With Assumption of $f(1) = c_1$, $f(4) = c_2$ and $f(5) = c_3$ and according to Eq. (13), we obtain that $f(3) = \alpha^2 / 6c_1$ and based on Eq. (12) as follows

$$\begin{aligned} &k = 0, \quad f(6) = \frac{\alpha^2}{10} c_2, \\ &k = 1, \quad f(7) = \frac{\alpha^2 c_3}{14} - \frac{2\alpha^2 c_1}{7!}, \\ &k = 2, \quad f(8) = \frac{144\alpha^4 c_2 + 2\alpha^2 \Lambda c_1}{8!}, \\ &k = 3, \quad f(9) = \frac{\left(-5\alpha^8 - 18\alpha^2 \Lambda\right) c_1 + 720\alpha^4 c_3}{9!}, \\ &k = 4, \quad f(10) = \frac{\left(10\alpha^4 \Lambda + 48\alpha^2 \Lambda\right) c_1 + 240\alpha^6 c_2}{10!}, \\ &\vdots \end{aligned} \tag{14}$$

By calculating up to the N_{th} term of $f(N)$ and substituting them into the upper limit of Eq. (13), the system of homogenous equations are arranged as follows

$$\begin{aligned} &f(0) + f(1) + f(2) + f(3) + f(4) \\ &+ \ldots + F(N) = 0, \\ &-\alpha^2 f(0) - \alpha^2 f(1) + (2-\alpha^2) f(2) \\ &+(6-\alpha^2) f(3) + (12-\alpha^2) f(4) + \ldots + f(N) = 0, \\ &-\alpha^2 f(1) - 2\alpha^2 f(2) + (6-3\alpha^2) f(3) \\ &+(24-4\alpha^2) f(4) + \ldots + f(N) = 0, \end{aligned} \tag{15}$$

Now, we may make the system of equations according to Eq. (15) in form of matrix equation as follows

$$\begin{aligned} &BC = 0; \\ &B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\ &C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad b_{ij} = func(\alpha, \Lambda) \end{aligned} \tag{16}$$

The condition where the determinant of B in Eq. (16) must disappear will provide an equation to specify $\Lambda = 72Dn^2$ under given value of α.

3.2 Solution by HPM

According to HPM (He 2005), we make the Homotopy of Eq. (10) as

$$H(f,p) = (1-p)(f^{(6)} - f_0^{(6)}) + p\begin{pmatrix} f^{(6)} - \alpha^6 f - 3\alpha^2 f^{(4)} \\ +3\alpha^4 f'' - \alpha^2\Lambda\left(2\eta^3 - 3\eta^2 + \eta\right) f \end{pmatrix} \quad (17)$$

Now, the general solution in form of power series has following form

$$f = f_0 + pf_1 + p^2 f_2 + p^3 f_3 + \ldots \quad (18)$$

First guess according to the considered boundary conditions is as follows

$$f_0 = C_1\eta\left(1 + \frac{1}{6}\alpha^2\eta^2\right) + C_2\eta^4 + C_3\eta^5 \quad (19)$$

By substituting f from Eq. (18) into Eq. (17) with assumption of $f_0^{(6)} = 0$, we have,

$$\begin{aligned} p^0 &: \ f_0^{(6)} = 0 \quad (20)\\ p^1 &: \ f_1^{(6)} - \alpha^6 f_0 - 3\alpha^2 f_0^{(4)} + 3\alpha^4 f_0'' - \alpha^2\Lambda\left(2\eta^3 - 3\eta^2 + \eta\right) f_0 = 0 \\ p^2 &: \ f_2^{(6)} - \alpha^6 f_1 - 3\alpha^2 f_1^{(4)} + 3\alpha^4 f_1'' - \alpha^2\Lambda\left(2\eta^3 - 3\eta^2 + \eta\right) f_1 = 0 \\ &\vdots \end{aligned}$$

We obtained the general solution of Eq. (18) by solving the set of differential Eq. (20). Afterward, the general solution corresponding to the boundary conditions (i.e. Eq. (11)) resulted to an equation to determine the solution for $\Lambda = 72Dn^2$ under the given value of α. Continued, semi-analytical results of DTM and HPM are presented and discussed.

4. RESULTS AND DISCUSSION

In the current section, we obtained the critical values of Dean number (Dn_C) by solving the eigenvalue problem governed on Dean flow stability using the DTM and the HPM. Our semi-analytical results are also compared with numerical solution. To achieve the numerical solution, Eq. (10) in association with Eq. (11) is solved using the spectral method, numerically. To this accomplishment, we expanded all variables with Chebyshev polynomials (Boyd 2002; Orszag 1971). In the current study, to assure the accuracy of our numerical results, we applied 120 truncation terms (Yanase *et al*. 2002; Yamamoto *et al.* 1994).
Critical Dean Numbers calculated by the HPM and DTM at several real constant wave numbers α under different number of truncated terms (i.e. N) are shown in Figs. 2 and 3, respectively. Figures 2 and 3 show our semi-analytical results are in very good accordance with our numerical solution, under all intended α. It is evident that the effectiveness of both semi-analytical methods is dependent on the value of α. Indeed, the number of truncated terms (N) needed to approach into our numerical data is increased by enhancement of the wave number (α). Figures 2 and 3 also show that at higher α, the HPM is more effective than the DTM. Indeed, HPM handles our intended eigenvalue problem with less computational cost.

For assessment of the accuracy of our solutions, semi-analytical results are compared with numerical solution by spectral method and we obtain the relative errors in tables 1 and 2 in respective for DTM and HPM. Table 1 and 2 show that both two semi-analytical methods of DTM and HPM are applicable for solving the eigenvalue problem occurring in linear hydrodynamics stability of Dean flow, whereas, the HPM has higher accuracy and effectiveness compared to DTM. Therefore, continued, the HPM is used to study the onset of Dean flow instability.

This fact may be related to more precise initial guess for the HPM compared to the DTM that resulted to faster and more accurate approximation for HPM. Moreover, due to implementation of Homotopy (i.e. small embedding parameter) in the HPM, it seems that the HPM compared to the DTM has a greater compatibility and consistency with linear hydrodynamic stability analysis which includes small disturbances.

Critical Dean numbers at different wave number α is presented in Fig. 4. As stated in pioneering works related to hydrodynamic stability analysis (Dean 1928; Walowit *et al*. 1964; Deka and Takhar 2004; Deka and Paul 2013; Ali *et al*. 1977), the minimum of critical Dean numbers ($Dn_{c,\min}$) and its corresponding α are the criteria for onset of instability. The results of Fig. 4 is achieved with HPM (N=17) where N is high enough to provides $e = \left(Dn_c(N) - Dn_c(N-1)\right).\left(Dn_c(N)\right)^{-1} < 10^{-6}$ (here, $Dn_c(N)$ is the critical Dean number calculated by the HPM under N number of truncated terms). Table 3 is also shows the $Dn_{c,\min}$ obtained from Fig. 4 compared with archival data presented in Ref. (Dean 1928; Walowit *et al*. 1964; Deka and Takhar 2004; Deka and Paul 2013; Ali *et al*. 1977). According to Table 3, we obtained the $Dn_c = 35.927$ corresponding to $\alpha = 3.952$ for the onset of Dean flow instability by using the HPM (N=17), which have acceptable accordance with archival data.

It's notable that, linear hydrodynamic stability analysis is founded on the linearized equation of perturbation and nonlinear terms are ignored in this analysis. However, due to accordance between the theory of linear hydrodynamic stability and the

experimental tests for Dean flow (Drazin and Reid 2004; Brewster *et al.* 1959), the nonlinearities are safe to be neglected for stability analysis of Dean flow.

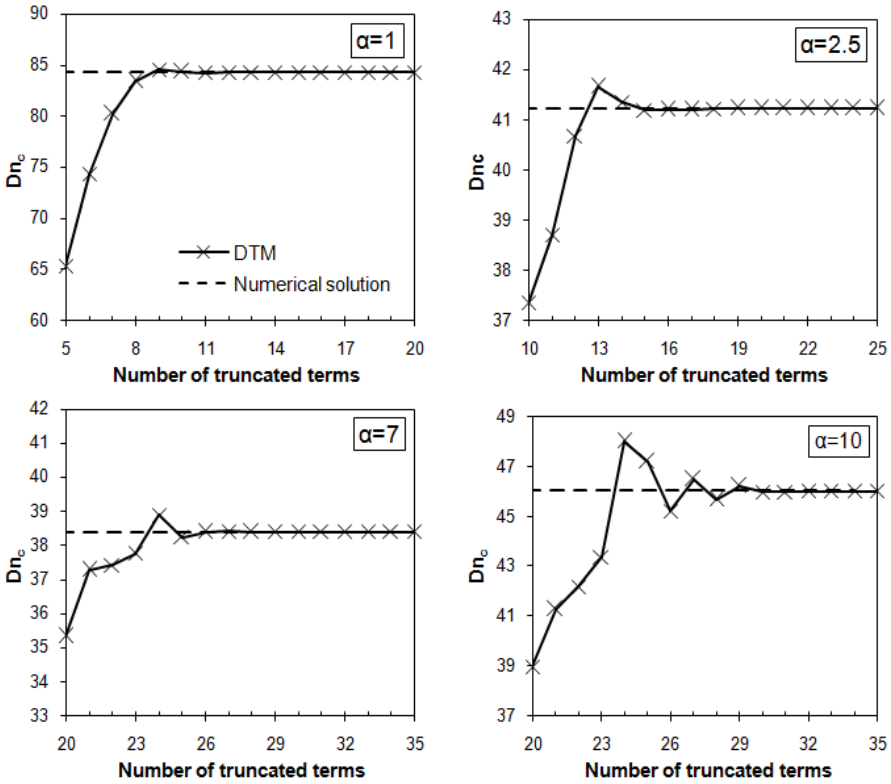

Fig. 2. Eigenvalue parameter, Dn_C , obtained by DTM at several α under different truncated terms.

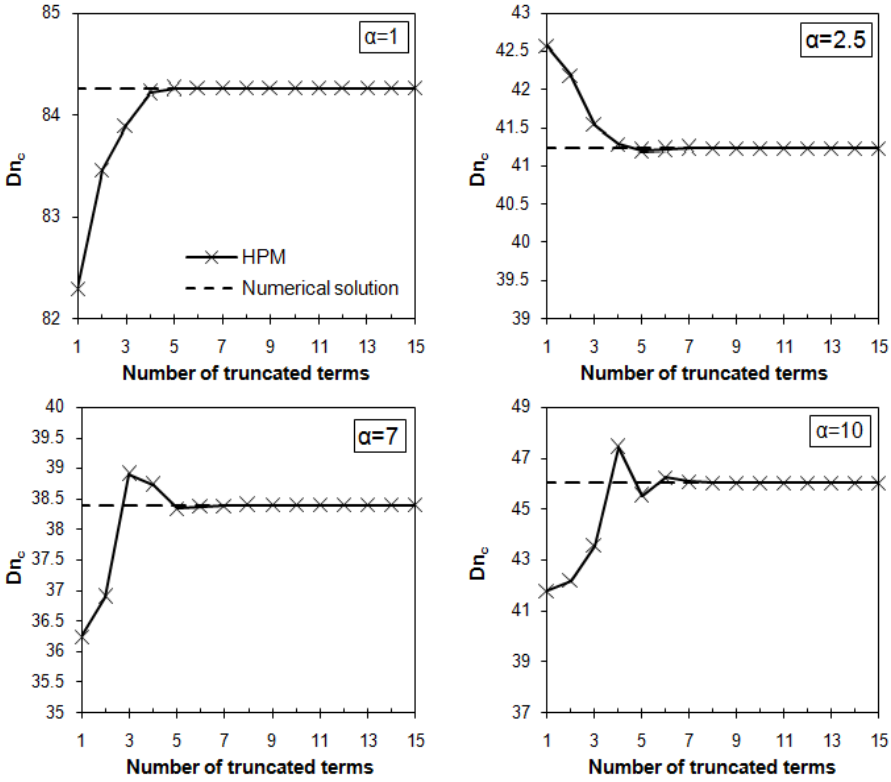

Fig. 3. Eigenvalue parameter, Dn_C , obtained by HPM at several α under different truncated terms.

Table 1 Eigenvalue parameter, Dn_C , achieved by DTM at several α under different number of truncated terms compared to our numerical solution.

α	DTM at several N				Numerical solution	Error (%) N=10	Error (%) N=35
	N=10	N=20	N=25	N=35			
1	84.33984575	84.26576920	84.26577080	84.26577318	84.26577614	0.088	3.51E-06
2.5	37.35080205	41.23199468	41.23240397	41.23242243	41.232429078	9.414	1.61E-05
7	25.19785888	35.33724587	38.24497484	38.39982479	38.399848201	34.380	6.09E-05
10	23.61180957	38.96940595	47.20980919	46.03309240	46.03320058	48.707	2.35E-04

Table 2 Eigenvalue parameter, Dn_C , achieved by HPM at several α under different number of truncated terms compared to our numerical solution.

α	HPM at several N				Numerical solution	Error (%) N=3	Error (%) N=15
	N=3	N=7	N=10	N=15			
1	83.90006267	84.26508431	84.26569937	84.26577560	84.26577614	0.434	6.36E-07
2.5	41.54188851	41.23604743	41.23260276	41.23243031	41.232429078	0.75	2.98E-06
7	38.90006713	38.38549856	38.39854903	38.39984629	38.399848201	1.302	4.97E-06
10	43.57502767	46.08751976	46.02972968	46.03319608	46.03320058	5.34	9.77E-06

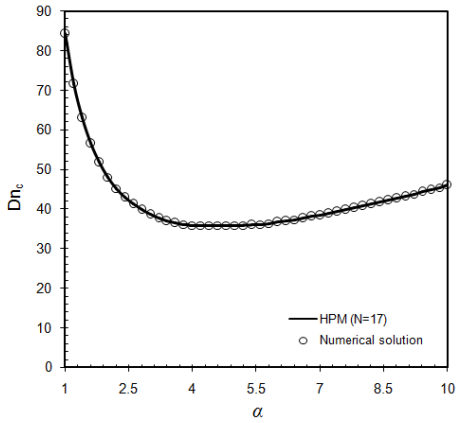

Fig. 4. Eigenvalue parameter of critical Dean number (Dn_c) at different wave number α .

Table 3 $Dn_{c,\min}$ and corresponding α from HPM (N=17) compared to archival data.

		Method of solution	α	$\Lambda_{c,\min} = 72Dn_{c,\min}^2$	$Dn_{c,\min}$
Present study		HPM (N=17)	3.952	92939.125	35.927
Archival data.	Dean 1928	Fourier expansion	3.954	93053	35.95
	Walowit *et al*. 1964	Galerkin method	3.96	92794.32	35.90
	Deka and Takhar 2004	Runge–Kutta Fehlberg method	3.96	92912	35.922
	Deka and Paul 2013	Differential transform method	3.950	92911.753	35.922
	Ali *et al*. 1977	Finite difference method	3.950	92782	35.897

5. CONCLUSION

We applied two semi-analytical methods of DTM and HPM to solve the eigenvalue problem governing on linear hydrodynamics stability of Dean Flow. Among the most important extracted findings in this study are the followings:

(1) At the similar number of truncated terms, the HPM gives more accurate solution compared to the DTM.

(2) We obtained that as the wave number increases, the number of truncated terms needed to approach into the numerical data are less for HPM compared to the DTM.

(3) We find the $Dn_c = 35.927$ corresponding to $\alpha = 3.952$ for the onset of Dean flow

instability which have acceptable accordance with archival data.

We deduced that HPM is an efficient semi-analytical approach in comparison with DTM to solve the eigenvalue problems governing on engineering sciences such as linear hydrodynamic stability of Dean flow.

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APPENDICES

APPENDIX A: BASIC CONCEPT OF DTM

Consider $v(x)$ as a semi-analytical function in a domain B, where $x = x_i$ indicate each point in B. In this case, the Taylor series expansion of $v(x)$ has following form:

$$v(x) = \sum_{k=0}^{\infty} \frac{(x - x_i)^k}{k!} \left[\frac{d^k v(x)}{dy^k} \right]_{x=x_i} \qquad \forall x \in B \quad \text{(A1)}$$

Where $x_i = 0$, Maclaurin series of $v(x)$ is

$$v(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k v(x)}{dx^k} \right]_{x=0} \qquad \forall x \in B \qquad \text{(A2)}$$

here, differential transformation for the function $v(x)$ has following form

$$V(k) = \sum_{k=0}^{\infty} \frac{H^k}{k!} \left[\frac{d^k v(x)}{dx^k} \right]_{x=0} \qquad \text{(A3)}$$

where, H is a constant, and the differential spectrum of $V(k)$ is located in distance of $x \in [0, H]$. Now, we have the inverse differential transform of $V(k)$ as follows,

$$v(x) = \sum_{k=0}^{\infty} \left(\frac{x}{H} \right)^k V(k), \qquad \text{(A4)}$$

If we consider $v(x)$ as a finite series, Eq. (A4) may be redefined as follows:

$$v(x) = \sum_{k=0}^{n} \left(\frac{x}{H} \right)^k V(k). \qquad \text{(A5)}$$

Table A1 shows the basic mathematical operations performed by DTM.

APPENDIX B: BASIC CONCEPT OF HPM

Consider a nonlinear differential equation with its boundary conditions as

$$L(v) + N(v) - g(r) = 0 \qquad r \in \Omega \qquad \text{(B1)}$$

$$BC: \qquad B(v, \partial v / \partial n) = 0 \qquad r \in \Gamma \qquad \text{(B2)}$$

where, $g(r)$ is an analytical function, L and N are linear differential operator and non-linear differential operator, respectively. Moreover, boundary operator is B, Γ is a boundary domain Ω and $\partial v / \partial n$ is a differentiation along the normal drawn outwards from Ω. By applying the Homotopy into Eq. (B1) , we have:

$$H(u, p) = L(u) - L(v_0) + pL(v_0) + p(N(u) - g(r)) = 0 \qquad \text{(B3)}$$

where

$$u(r,p):\Omega\times[0,1]\to R, \tag{B4}$$

In Eq. (B3), initial approximation is introduced via v_0 that is in accordance with boundary condition. In addition, embedding parameter is shown by $p\in[0,1]$. By redefinition of the solution of Eq. (B3) in form of power series (i.e. p), we have ,

$$u = u_0 + pu_1 + p^2u_2 + \ldots \tag{B5}$$

Now, approximated solution has following from

$$v = \lim_{m\to 1} u = u_0 + u_1 + u_2 + \ldots \tag{B6}$$

Table A1 Fundamental mathematical operations in DTM

Original Function	Transformed Function
$v(x)=\alpha f(x)\pm\beta g(x)$	$V(k)=\alpha F(x)\pm\beta G(x)$
$v(x)=f(x)g(x)$	$V(k)=\sum_{k_1}^{k}F(k_1)G(k-k_1)$
$v(x)=\dfrac{d^m f(x)}{dx^m}$	$V(k)=(k+1)(k+2)$ $\ldots(k+m)F(k+m)$
$v(x)=\int_{x_0}^{x} f(x)dx$	$V(k)=\dfrac{F(k-1)}{k},k\geq 1$
$v(x)=x^m$	$V(k)=\delta(k-m),$ $\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$
$v(x)=\exp(x)$	$V(k)=\dfrac{\lambda^k}{k!}$