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Flutter of a Membrane in a Stagnation Flow

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ABSTRACT

An analytical solution is given to investigate the vibrations of a membrane under the effect of an incoming fluid flow perpendicular to it. The membrane is located at the stagnation point of the flow and is of finite width but infinite length. A rigid wall extends through the finite width of the membrane to infinity. The flow is considered to be a small perturbation on the two dimensional potential stagnation flow solution due to the vibrations of the membrane, and the membrane is modeled by the linear vibration equation. The resulting coupled problem is solved by a Galerkin procedure and the eigenvalue equation relating the membrane frequency to the other parameters is derived.

Keywords: Flutter velocity; Membrane; Stability; Stagnation point flow; Vibration.

1. INTRODUCTION

Membrane structures are widely encountered in many forms including tympanic membrane in the human ear, lightweight (inflatable) civil and aerospace structures, paper manufacturing processes, several types of pumps, sails and parachutes. Naturally, these structures will be acted upon by forces due to the fluid in which they are situated. The case in which the fluid is more or less stagnant is of interest to the biomechanics of the ear drum for example (deBoer and Viergever, 1983). In many cases, the fluid flow is essentially parallel to the undisturbed membrane; in this case, small disturbance theory familiar from thin airfoil aerodynamics has been succesfully applied to a range of problems.

Newman and Paidoussis (1991) considered the stability of a membrane similar to present study, but under an approximately parallel flow. They used small disturbace theory and investigated the stability behavior with respect to incidence angle and membrane tension. Newman and Tse (1980), again using small disturbance or thin airfoil theory, computed the flow around an airfoil made up of two curved membranes. Newman (1987) using similar techniques gave a general account of the aerodynamics of membranes. Newman and Low (1984) reported about experiments on sails. Kunieda (1975) considered vibrations of curved membranes under parallel flow. Szygulski (1997) considered the stability of a membrane bounded by rigid walls using finite and boundary element methods. Yamaguchi *et al.* (2000), using a distributed vortex approach, computed the stability behavior of membranes in high speed flow. Huang (2001) investigated the vibration behavior of a membrane which forms the part of the boundary of a viscous Poiseuille flow. Pan *et al.* (2001) developed a model for a valveles micropump involving plate theory, while Watanabe *et al.* (2002) considered paper flutter using both potential flow and viscous flow approaches. In most of these studies, except (Pan *et al.* 2001), the fluid flow is predominantly parallel to the membrane surface.

One aspect of the attempt to understand the dynamics of membrane structures under fluid flow that seems to be missing is the case in which the flow is perpendicular to the membrane. This kind of study may be important in inflatable structures and sails or conceivably some kind of biomechanical problem. Another significant application of membrane-fluid interactions is energy harvesting from mechanical vibrations using piezoelectric materials (Li *et al.* 2014). A conventional fluttering device is arranged in parallel with the flow direction, but experiments by Li *et al.* (2011) show that the cross flow architecture could help to increase the performance of such devices. This study may be used in the theoretical investigation of piezo-electric energy harvesting employing vibrations of thin membrane structures under the effect of an incoming flow. At any rate, the proposed problem can be considered as one of the basic problems of membrane-fluid interactions.

In this paper, a membrane in the shape of an infinite strip placed symmetrically at the stagnation point of an incoming incompressible irrotational flow has been studied. The membrane vibrates due to disturbances in the flow; the conditions under which these vibrations lead to instability will be derived.

The choice of potential flow for the investigation of flutter phenomena in general stems from the fact that it is the normal forces, i.e., pressure, rather than the shear forces parallel to the membrane surface that cause the membrane to vibrate. Therefore, the viscous or shear forces are of secondary importance and can be neglected. Inviscid flows naturally tend to be potential or irrotational since viscosity is mainly responsible for imparting rotation on infinitesimal fluid elements. Also, the assumption is made in this study that the flow velocity is not very high so that the incompressibility assumption holds. As a result, the flow field is governed by the Laplace's equation.

2. PROBLEM FORMULATION AND SOLUTION

A schematic of the problem is shown in Fig. 1. The membrane occupies $-b \le x \le b$ and is of infinite length in z-direction. The stagnation point of the incoming two dimensional incompressible potential flow is the origin. As is wellknown (Currie, 1974), the potential flow around a stagnation point on a rigid wall is approximately given by the velocity potential

$$\phi = \frac{1}{2}U(x^2 - y^2) \tag{1}$$

satisfying the two dimensional Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2}$$

U is a constant with dimension $(\text{time})^{-1}$ .This can be extended to the case in Fig. 1 , where part of the rigid wall is replaced by the membrane, by assuming that the flow differs from Eq.(1) by a small amount $\phi_1(x, y, t)$, the perturbation potential, which is time-dependent since the vibrations of the membrane disturbs the original steady flow given by Eq.(1),

$$\phi(x, y, t) = \frac{1}{2}U(x^2 - y^2) + \phi_1(x, y, t) \tag{3}$$

It should be noted here that the unsteady potential incompressible flow also satisfies the Laplace's equation. Substituting Eq.(3) in Eq.(2) gives

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \tag{4}$$

The governing partial differential equation for the tranverse membrane vibrations is

$$T\frac{\partial^2 v}{\partial x^2} - \rho_m \frac{\partial^2 v}{\partial t^2} - p = 0 \tag{5}$$

$v(x, t)$ is the displacement of the membrane, ρ_m is the mass per unit area of the membrane, T is tension per unit lenght, and p is the fluid pressure on the membrane computed from the unsteady Bernoulli equation

$$p = -\frac{1}{2}\rho\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] - \rho\frac{\partial \phi}{\partial t} \tag{6}$$

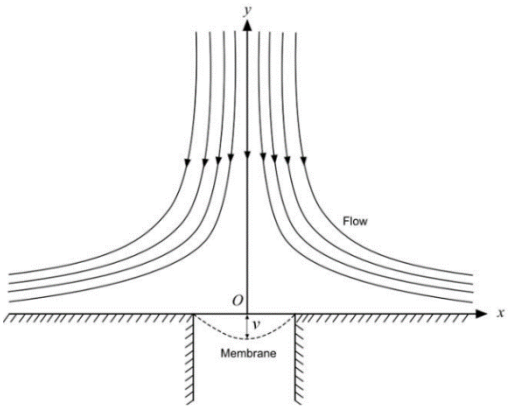

Fig. 1. Two dimensional stagnation point flow near the membrane.

ρ is the density of the fluid. Ignoring the nonlinear terms in terms of ϕ_1 and taking the membrane to be approximately at $y = 0$, the pressure on the membrane becomes

$$p = -\frac{1}{2}\rho U^2 x^2 - \rho\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi}{\partial t}\right)_{y=0} \tag{8}$$

The first term causes a static deflection of the membrane. The other term is time-dependent and causes vibrations in the membrane which are of interest. Therefore, we use the second term as the forcing in the vibration equation Eq. (5),

$$T\frac{\partial^2 v}{\partial x^2} - \rho_m \frac{\partial^2 v}{\partial t^2} - p_1 = 0 \tag{8}$$

with

$$p_1 = -\rho\left(Ux\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_1}{\partial t}\right)_{y=0} \tag{9}$$

The boundary conditions on the membrane are that it is fixed at the sides :

$$v = 0 \quad \text{at} \quad x = \pm b \tag{10}$$

On the membrane, the normal components of the fluid and membrane velocities should be the same; again taking the membrane to be at $y = 0$ and ignoring nonlinear terms, this gives

$$\left.\frac{\partial \phi_1}{\partial x}\right|_{y=0} = \frac{\partial v}{\partial t} + Ux\frac{\partial v}{\partial x} = f(x, t) \quad \text{for} \quad |x| \le b \tag{11}$$

$$\left.\frac{\partial \phi_1}{\partial x}\right|_{y=0} = 0 \quad \text{for} \quad |x| \le b \tag{12}$$

where, the middle expression in Eq.(11) is given the name $f(x,t)$ for brevity. Finally, ϕ_1 should fall to zero fast enough at infinity. The solution of Eq.(4) with Eq.(12) is

$$\phi_1(x,y,z) = \frac{1}{2\pi}\int_{-b}^{b} \ln\left[(x-\xi)^2 + y^2\right] f(\xi,t)\, d\xi \tag{13}$$

To investigate the vibrations of the membrane, the membrane displacement is taken as

$$v(x,t) = V(x) e^{i\omega t} \tag{14}$$

Since the problem is linear, each variable in the problem is assumed to be harmonic (i.e., real or imaginary parts are taken in Eq. (14), for example). Different frequencies do not effect each other, and only one parametric (angular) frequency ω can be considered. Similarly, the pressure becomes

$$p_1(x,t) = P(x) e^{i\omega t} \tag{15}$$

where

$$P = -\frac{\rho}{\pi}\int_{-\infty}^{\infty}\left[\frac{Ux}{x-\xi} - i\omega \ln|x-\xi|\right] F(\xi)\, d\xi \tag{16}$$

$$F(x) = i\omega V(x) + UxV'(x) \tag{17}$$

And $V(x)$ satisfies

$$T\frac{d^2V}{dx^2} + \rho_m \omega^2 V + -\frac{\rho}{\pi}\int_{-\infty}^{\infty}\left[\frac{Ux}{x-\xi} - i\omega \ln|x-\xi|\right] F(\xi)\, d\xi = 0 \tag{18}$$

$$V = 0 \qquad \text{at} \qquad x = \pm b \tag{19}$$

Equations Eq. (17) and Eq. (18) constitute an integro-differential eigenvalue problem for $V(x)$ with the eigenvalue ω.

$$V(x) = \sum_{n=1}^{N} a_n \sin\frac{n\pi x}{b} \tag{20}$$

This form satisfies the boundary conditions (19). In the Galerkin method, after substituting the assumed solution form, Eq. (20), into the governing equation, the inner product of the result (called the residual) and the base functions (the sine functions in Eq. (20) in this case) are equated to zero to give a set of (infinite) algebraic equations. Substituting

Eq. (20) into (18), multiplying by $\sin\left(\frac{m\pi x}{b}\right)$ and integrating between $-b$ and b, we obtain

$$\left(-T\left(\frac{m\pi}{b}\right)^2 + \rho_p h_p \omega^2\right) b a_{m+} \frac{\rho}{\pi}\sum_{n=1}^{N} a_n \left\{ i\omega(i\omega + U) b^2 A_{nm} + i\omega n\pi U b^2 B_{nm} + U(i\omega + U) b^2 C_{nm} + n\pi U^2 b^2 D_{nm} \right\} = 0 \tag{21}$$

for $m = 1, 2, \ldots . N$. Here

$$A_{nm} = \int_{-1}^{1}\int_{-1}^{1} \ln|v-u| \sin(n\pi u)\sin(m\pi v)\, du\, dv \tag{22}$$

$$B_{nm} = \int_{-1}^{1}\int_{-1}^{1} u \ln|v-u| \cos(n\pi u)\sin(m\pi v)\, du\, dv \tag{23}$$

$$B_{nm} = \int_{-1}^{1}\int_{-1}^{1} u \ln|v-u| \cos(n\pi u)\sin(m\pi v)\, du\, dv \tag{24}$$

$$D_{nm} = \int_{-1}^{1}\int_{-1}^{1} \frac{uv}{v-u} \cos(n\pi u)\sin(m\pi v)\, du\, dv \tag{25}$$

These integrals can be evaluated in terms of sine and cosine integrals

$$Si(x) = -\int_0^x \frac{\sin t}{t} dt \tag{26}$$

$$Ci(x) = -\int_x^{\infty} \frac{\cos t}{t} d \tag{27}$$

The resulting expressions are long; they are written in the appendix for reference. Eq. (21) is a linear homogeneous system of algebraic equations; the determinant of the coefficients give the characteristic eigenvalue equation for ω. We use the following non-dimensional quantities.

$$\bar{T} = \frac{T\pi^2}{\rho b} \tag{28}$$

$$k = \frac{\omega}{U} \tag{29}$$

$$\mu = \frac{\rho_m h_m}{\rho b} \tag{30}$$

μ is basically the mass ratio between the membrane material and fluid. With these definitions, Eq.(21) becomes

$$\left(-\frac{\bar{T}}{(Ub)^2} m^4 + \mu k^2\right) a_m + \frac{1}{n}\sum_{n=1}^{N}\{ik(ik+1)A_{nm} + ikn\pi B_{nm} + (ik+1)C_{nm} + n\pi D_{nm}\} a_n = 0 \tag{31}$$

Note that U has the dimension of velocity per length while Ub has the dimension of velocity. We will refer to Ub as the velocity.

3. RESULTS AND DISCUSSION

Eqs. (31) are a linear system of homogeneous algebraic equations for the expansion coefficients an . The determinant of this system is a_n algebraic equation of the form

$$f\left(\mu, k, \frac{\bar{T}}{(Ub)^2}\right) = 0 \tag{32}$$

For given values of Ub and μ, Eq. (32) is to be solved for the complex eigenvalue k. If the imaginary part of k is negative the motion is stable, since $v(x,t) = V(x)e^{iUkt}$. The value of Ub at which $\mathrm{Im}(k)$ first becomes zero is the flutter velocity above which there are always unstable modes present.

The accuracy of the method increases as the number or modes N increases. In general, Eq.(32) has to be solved numerically; but for $N = 1$, we can obtain an explicit expression for flutter velocity:

$$(Ub)_{fl} = \sqrt{\frac{\bar{T}}{D_{11}}} = 3,6689\sqrt{\frac{\bar{T}}{\rho b}} \tag{33}$$

This simply follows from Eq.(31), for the case $m = n = 1 = N$, by noting that $A_{11} = B_{11} = C_{11} = 0$, and setting the discriminant of the resulting quadratic equation for k to zero.

Taking 3 to 5 terms in the expansion gives adequate accuracy. It seems that the flutter velocity computed from Eq. (33) is about 15% higher than the accurate result as long as $\bar{T}$ is less than 5000. Fig. 2 shows both results for the flutter velocity as a function of $\bar{T}$. Another interesting result is the fact that the flutter velocity is independent of the mass ratio μ although it is present in Eq.(32). This is explicitly seen from Eq. (33) and it is also confirmed by computations with higher number of modes.

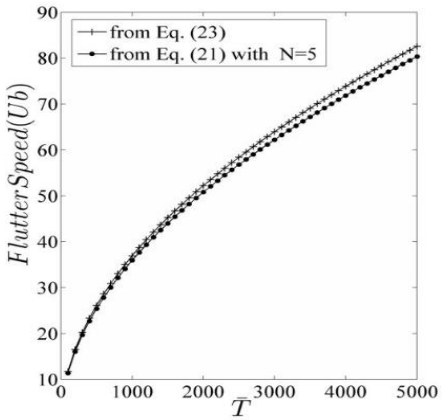

Fig. 2. Variation of the flutter velocity with respect to $\bar{T}$, (+) from Eq.(23), (•) from Eq.(21) with N=5.

Flutter velocity increases with the increase of the tension force on the membrane as should be expected; and can be roughly estimated by Eq.(33), which states that $(Ub)_{fl}$ is proportional to the square root of the tension force.

4. CONCLUSION

An analytical solution was derived that allows computation of the critical flutter velocity of an incompressible potential flow towards a membrane placed at the stagnation point. The end result basically can be summarized with Eq.(33) . The flutter velocity is proportional to the square-root of the tension force on the membrane. If the formula is modified slightly as mentioned before,

$$(Ub)_{fl} = 3,0966\sqrt{\frac{\bar{T}}{\rho b}} \tag{34}$$

this gives the flutter velocity in the linearized approximation quite closely.

The result presented in Eq. (34) is simple, elegant, and might be very useful in practice. The engineer can change the parameters of the problem, i.e., physical constants of the material and the tension in the membrane, and quickly infer whether the design under consideration will lead to unstable oscillations, or the safety margin before the danger of instability.

A APPENDIX

Explicit expressions for A_{nm}, B_{nm}, C_{nm} and D_{nm} in terms of the sine and cosine integrals:

$$A_{nm} = \begin{cases} -\dfrac{2(\gamma - Ci(2\pi n) + \ln(2\pi n) + n\pi Si(2\pi n))}{n^2\pi^2} & ,m = n \\ \dfrac{2(-1)^{n+m}}{m(n-m)n(m+n)\pi^2}((m-n)(m+n)\gamma \\ +n^2(Ci(2m\pi) - \ln(2m\pi)) + \\ m^2(-Ci(2n\pi) + \ln(2n\pi))) & ,m \neq n \end{cases}$$

$$B_{nm} = \begin{cases} -\dfrac{2(\gamma - Ci(2\pi n) + \ln(2\pi n) + n\pi Si(2\pi n))}{n^3\pi^3} & ,m = n \\ \dfrac{1}{mn^2(m^2-n^2)^2\pi^3}2(-1)^{m+n} \\ ((m-n)(m+n) \\ (m^2(-1+\gamma) - n^2\gamma) - n^2\gamma) - \\ n^2(m^2+n^2)Ci(2m\pi) - \\ (m^4 - 3m^2n^2)Ci(2n\pi) - \\ m^2n^2\ln 2 + m^2(m-n)(m+n) \\ \cos(2m\pi)(1+\ln 2) + m^2n^2\ln(m) \\ -3m^2n^2\ln(n) - 2m^2n^2\ln(\pi) + \\ n^4\ln(2m\pi) + m^4\ln(n\pi) + \\ m(m-n)n(m+n)\pi(nSi(2m\pi) \\ +mSi(2n\pi))) & ,m \neq n \end{cases}$$

$$C_{nm}=\begin{cases}0 & ,m=n\\ \frac{1}{(m^2-n^2)^2\pi^2}2(-1)^{m+n} & \\ (m(-2(m^2+3n^2)+ & \\ (m^2-n^2)^2\pi^2)Ci(2m\pi)+ & \\ m(2m^2+6n^2- & \\ (m^2-n^2)^2\pi^2)Ci(2n\pi)+ & \\ m(-2(m^2+3n^2)+ & \\ (m^2-n^2)^2\pi^2)(-\ln(m)+ & \\ \ln(n))+2(m^4-n^4)\pi Si(2m\pi) & \\ +4m(m-n)n(m+n)\pi Si(2n\pi)) & ,m\neq n\end{cases}$$

$$D_{nm}\begin{cases}\frac{2}{3}(-\frac{1}{n\pi}+Si(2n\pi)) & ,m=n\\ \frac{1}{(m^2-n^2)^3\pi^3}2(-a)^{m+n} & \\ (m(-2(m^2+3n^2)+ & \\ (m^2-n^2)^2\pi^2)Ci(2m\pi)+ & \\ +m(2m^2+6n^2- & \\ (m^2-n^2)^2\pi^2)Ci(2n\pi) & \\ +m(-2(m^2+3n^2)+ & \\ (m^2-n'^2)^2\pi^2)(-\ln(m)+ & \\ \ln(n)+2(m^4-n^4)\pi Si(2m\pi) & \\ +4m(m-n)n(m+n)\pi Si(2n\pi)) & ,m\neq n\end{cases}$$

REFERENCES

Currie, I. (1974). Fundamental Mechanics of Fluids. *McGraw-Hill Book Company*.

deBoer, E. and M. A. Viergever (1983). Mechanics of hearing. In *Proceedings of the IU-TAM/ICA symposium, Netherlands*, pp. 13-15. Delf University of Technology.

Huang, L. (2001). Viscous Flutter of a Finite Elastic Membrabe in Poiseuille Flow. *Journal of Fluid and Structures* 15(7), 1061–1088.

Kunieda, H. (1975). Flutter of hanging roofs and curved membrane roofs. *Journal of Solids and Structures* 11(4), 477–492.

Li, H., C. Tian and Z. D. Deng (2014). Energy harvesting from low frequency applications using piezoelectric materials. *Applied Physics Reviews* 1(4), 041301–1 – 041301–20.

Li, S., J. Yuan and H. Lipson (2011).Ambient wind energy harvesting using cross-flow fluttering. *Journal of Applied Physics* 109(2), 026104–1 – 026104–3.

Newman, B. C. and M. P. Paidoussis (1991).The stability of two-dimensional membranes in streaming flow. *Journal of Fluid Mechanics* (5), 443–454.

Newman, B. G. (1987). Aerodynamic theory for membranes and sails. *Progrees in Aerospace Sciences* 24(1), 1–27.

Newman, B. G. and H. T. Low (1984). Two-dimensional impervious sails: experimental results compared with theory. *Journal of Fluid Mechanics* 144, 445–462.

Newman, B. G. and M. C. Tse (1980). Flow past a thin, inflated lenticular aerofoil. *Journal of Fluid Mechanics* 100(4), 673–689.

Pan, L. S., T. Y. Ng, G. R. Liu, K. Y. Lam (2001). Analytical solutions for the dynamic analysis of a valveless micropump a fluid membrane coupling stu290dy. *Sensors and Actuators A: Physical* 93(2), 173–181.

Szygulski, R. (1997). Numerical analysis of membrane stability in air flow. *Journal of Sound and Vibration* 201(3), 281–292.

Watanabe, Y., K. Isogai, S. Suzuki and M. Sugihara (2002). A Theoretical Study of Paper Flutter. *Journal of Fluids and Structures* 16(4), 543 – 560.

Yamaguchi, N., K. Yokota and Y. Tsujimoto (2000). Flutter Limits and Behaviors of Flexible Thin Sheet in High-Speed Flow-1: Analytical Method for Prediction of the Sheet Behavior. *Journal of Fluids Engineering* 122, 65–73.