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An Efficient Radial Basis Function Meshfree Local Petrov–Galerkin Method for Modeling the Unidirectional Fully Developed Fluid Flow

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ABSTRACT

In hydrodynamics the applications range of incompressible flows is very wide. In this study, a robust, high order modeling approach is introduced, based on the MLPG meshfree method -based radial basis functions (RBF- MLPG) method, for solving the incompressible flow field. In other words a MLPG meshfree method based on an interpolation function is presented to solve the 2- D unsteady incompressible fully developed fluid flows. This meshfree method is based on the quartic (4^{th} order) spline. The method is then compared against the Finite Element Method on a test case of unidirectional fully developed incompressible fluid. The performance of this weight function proved that the quartic (4th order) spline gains the highest accuracy, convergence and efficiency. Finally, it can be concluded that the presented method is formidable for simulating fluid dynamics.

Keywords: MQ-RBF; Shape and Weight Functions; Unsteady flow; MLPG meshfree method; 2D Fluid Flow.

NOMENCLATURE

C	shape parameter	α_s	support domain
d_c	characteristic length	Δt	time step
n	number of nodes	μ	dynamic viscosity
r_Q	radius of the quartic spline weight function	Γ_Q	boundary of the quartic (4^{th} order) spline
t	time	Ω_Q	quartic (4^{th} order) spline domain
r_s	radius of the support domain	Ω_s	Support interpolation
$u^h(x,y)$	Radial Basis Function interpolation	ρ	density of the fluid
V_Z	velocity	$\Phi(x)$	Radial Basis Function
W	quartic (4^{th} order) spline	α	temporal weighting factor
α_Q	size of quadrature domain	$-\partial P/\partial z$	pressure gradient

1. INTRODUCTION

Fluid mechanics plays an important role in scientific research in civil and mechanic engineering. Scientists have been doing research on new techniques to solve differential equations. Same as the other disciplines, Computational Fluid Dynamics is developed by numerical analysis. Numerical analysis means to find the quantitative results of the problem by the mathematical method. Numerical methods are widely used in fluid dynamics. In the recent years, many numerical methods have been proposed and developed for analyses of fluid mechanics. Computational methods have used in fluid mechanics subjects for analyses of engineering problems (Chen *et al.*, 2015; Harichandan and Roy, 2012; Shirani *et al.*, 2011; Bunsri *et al.*, 2008). Also, Seyedashraf and Akhtari (2017) presented a new numerical scheme based on the Finite Element Method with a total-variation-diminishing. In addition, Xiaoming *et al.* (2019) combined the finite element method hybrid approach with RBFs to solve three-dimensional electromagnetics problems. Development of computational methods in engineering problems is very important. At present, research on meshfree methods has become one of the hottest research fields in engineering problems. meshfree methods like RKPM meshfree method (Tang *et al.*, 2018), EFG meshfree method (Cheng *et al.*, 2017), MLPG meshfree method (Saeedpanah, 2017), NEM meshfree method (Zhang *et al.*, 2013), DLSM meshfree method (Afshar and Shobeyri 2010). Using least-square meshfree method, DLSM meshfree method was proposed for solving fluid mechanics problems by Afshar and Shobeyri (2010).

One of the strongest of these methods is MLPG meshfree method which has been examined via many engineering problems. Atluri and Zhu expressed the MLPG meshfree method in 1998 and Atluri and Shen (2002) extended this method widely.

The RBF method was introduced as an interpolation scheme in the 1970s and later used to numerically approximate solutions to PDEs in the 1990s. The potential applications of RBFs grewas a result of using RBFs instead of polynomials. RBFs have been applied in the last decade to a wide range of PDEs that arise in, for example, fluid mechanics, wave motions, groundwater flow, astrophysics, geosciences, mathematical biology, elasticity, and flame propagation (Bayona *et al.*, 2019; Flyer and Fornberg, 2011).

The MQ-RBF produces a discretization for scattered nodes. It is a straightforward approach for spatial node refinement and it provides adjustable control over the order of accuracy. The MQ-RBF is at least competitive, and sometimes superior, to FDM and the FEM in some key applications in the fluid dynamics. Yet, there are still relatively few computational scientists who have had first-hand experience with either RBF or RBF-meshfree discretizations for large-scale PDE applications. A contributing factor might be that articles in the area still often focus on small-scale PDE (or ODE) test problems.

The Radial Basis Function (RBF) is a considerable feature which aid to solve flow domain. Recently, radial basis functions have been used in computational intelligence systems. For example, Seyedashraf *et al.* (2018) presented a new methodology based on the computational intelligence system for modeling the dam-break flow. Also, Wu *et al.* (2019) used a radial basis function to model wave shoaling. In addition, Zheng *et al.* (2017) proposed a radial basis function meshfree method.

In the present article, MLPG meshfree method has been extended for fluid flow analysis of two dimensional. In other word, a robust high-order MLPG meshfree based RBF method (RBF- MLPG) was developed to solve the 2- D unsteady incompressible fully developed fluid flows. In addition, a MLPG meshfree method with MQ-RBF interpolation is presented for solving two-dimensional unsteady incompressible fully developed fluid flow. Furthermore, this work presents a numerical scheme based on a quartic (4th order) spline as the weight function over a quadrature domain. Subsequently, the accuracy, convergence and efficiency of the weight function are investigated. Employed as weight function, shape functions have the biggest contribution in the solution in terms of accuracy, convergence and efficiency. Therefore, in this research two important novelties are introduced. A robust high-order MLPG meshfree based RBF method (RBF- MLPG) was developed to solve the 2- D unsteady incompressible fully developed fluid flows. In addition, a MLPG meshfree method with quartic (4^{th} order) spline as the weight function is presented for solving two-dimensional unsteady incompressible fully developed fluid flow.

2. MQ-RBF INTERPOLATION

The RBF method is generalizable to higher dimensions. It also has the advantage of being mesh independent, allowing the rendition of very complicated domains and boundaries and an improved accuracy with a node refinement (Flyer *et al.*, 2016; Bayona *et al.*, 2017; Bayona and Kindelan, 2013).

Hardy (1971) introduced the RBF methodology. For multiquadric RBFs in Micchelli (1986), rapidly accelerated the development of RBFs for solving PDEs. Extensive analysis of RBFs on infinite lattices was also carried out in the 1990s. Kansa (1990a, b) introduced the idea of analytically differentiating (spatial) RBF interpolants and thereby obtained a novel numerical approach for solving both steady-state and time dependent PDEs.

Using RBF expression, u(x), is defined as:

$$u^h(\mathrm{x},\mathrm{x}_Q) = \sum_{i=1}^{n} R_i(\mathrm{x})\bar{a}_i(\mathrm{x}_Q) = \mathrm{R}^T(\mathrm{x})\mathrm{a}(\mathrm{x}_Q) \tag{1}$$

where $R_i(x)$ is the RBF, n is the number of nodes, and $\bar{a}_i(X_Q)$ are the parameters for $R_i(x)$. Vectors are as follows:

$$\bar{a} = [\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_n]^T \tag{2}$$

$$\mathrm{R}^T = [R_1(\mathrm{x}), R_2(\mathrm{x}), R_3(\mathrm{x}), \ldots, R_n(\mathrm{x})]^T \tag{3}$$

The parameters for $\bar{a}_i(X_Q)$ can be obtained as follows:

$$u_k = u(x_k, y_k) = \sum_{i=1}^{n} \bar{a}_i(\mathrm{x}_Q) R_i(x_k, y_k), \quad k = 1,2,3,\ldots,n \tag{4}$$

which can be written as:

$$R_Q a = U_s \tag{5}$$

where $U_s = [u_1, u_2, u_3, \ldots, u_n]$ and R_Q is as follows:

$$R_Q = R_Q^T = \begin{bmatrix} R_1(r_1) & R_2(r_1) & \ldots & R_n(r_1) \\ R_2(r_2) & R_2(r_2) & \ldots & R_n(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_1(r_n) & R_2(r_n) & \ldots & R_n(r_n) \end{bmatrix}$$

(6)

Consequently, the interpolation is as follows:

$$u^h(x) = R^T(x) R_Q^{-1} U_s = \Phi(x) U_s \tag{7}$$

where shape functions is defined as:

$$\Phi(x) = [R_1(x), R_2(x), \ldots, R_k(x), \ldots, R_n(x)] R_Q^{-1} = [\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x), \ldots, \varphi_n(x)] \tag{8}$$

in which

$$\varphi_k(x) = \sum_{i=1}^{n} R_i(x) S_{ik}^a \tag{9}$$

and S_{ik}^a is (i,k) element of R_Q^{-1}. A multi Quadrics Radial Basis Function is expressed as: (Hardy, 1990; Frank, 1982; Kansa, 1990a, b; Liu and Gu, 2001):

$$R_i(x, y) = (r_i^2 + (\alpha_c d_c)^2)^q \quad \alpha_c \geq 0 \quad \text{(MQ)} \tag{10}$$

where q and α_c are shape parameters. Figures 1 to 6 show shape functions.

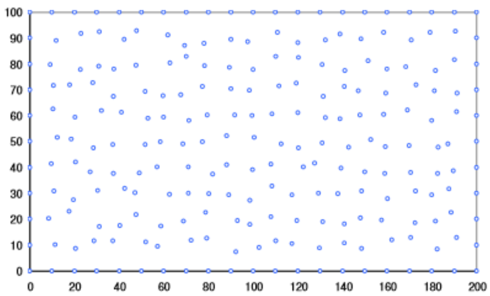

Fig. 1. Domain (231 nodes).

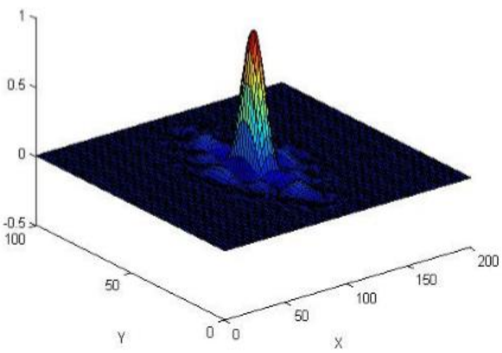

Fig. 2. RBF interpolation.

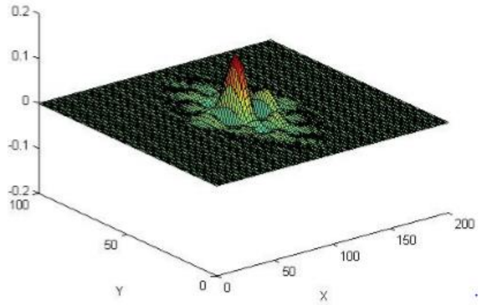

Fig. 3. Derivatives of RBF interpolation (x direction).

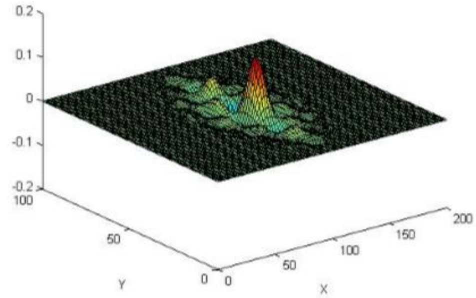

Fig. 4. Derivatives of RBF interpolation (y direction).

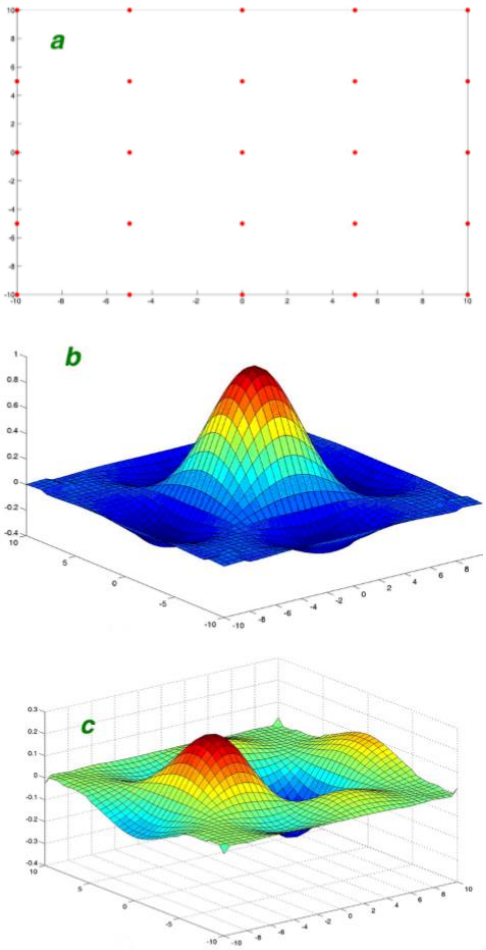

Fig. 5. RBF interpolation (25 nodes), a) Domain, b) RBF interpolation, c) Derivatives of RBF interpolation (x direction), d) Derivatives of RBF interpolation (y direction).

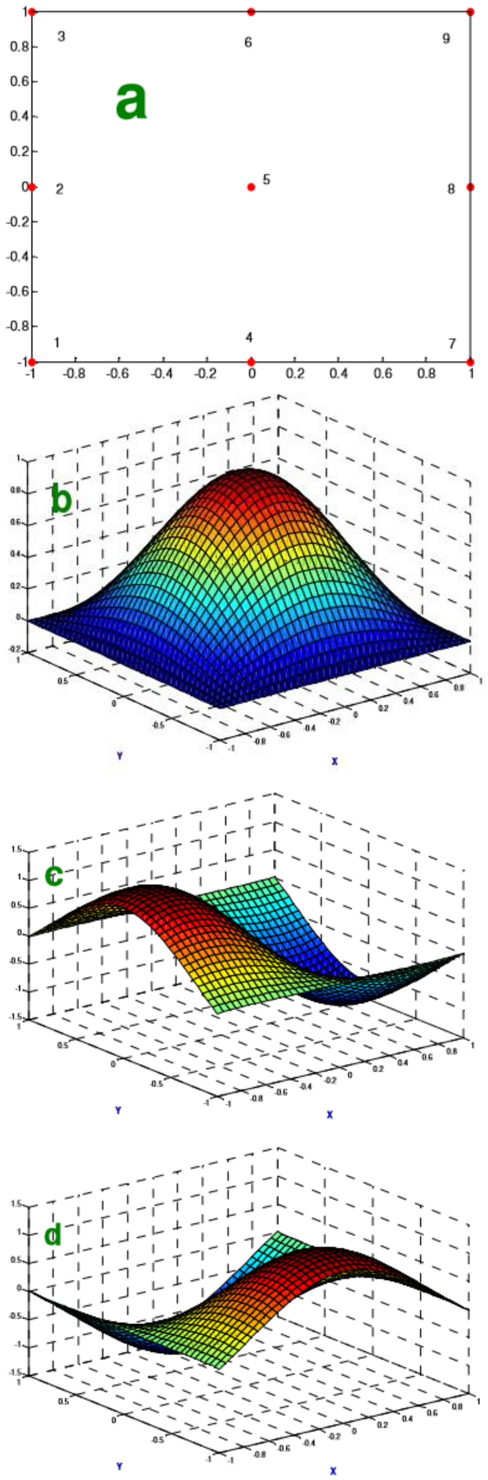

Fig. 6 RBF interpolation (9 nodes), a) Domain, b) RBF interpolation, c) Derivatives of RBF interpolation (x direction), d) Derivatives of RBF interpolation (y direction).

3. GOVERNING EQUATIONS

Figure 7 shows a Cross-section of a unidirectional fully developed incompressible fluid. The axis of channel is described by z coordinate as shown in Fig. 7. Therefore the Governing equations can be written as:

$$\mu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}\right)-\frac{dP}{dz}=\rho\frac{\partial V_z}{\partial t} \tag{11}$$

The Initial condition is:

$$at\ t=0, V_z=\text{Initial velocity}\ (V_{z\ \text{ini}}),\ \text{in}\ \Omega \tag{12}$$

The boundary conditions which are of essential type are as below:

$$\begin{aligned} V_z &= V_{z\,A_1}=0\\ V_z &= V_{z\,A_2}=0\\ V_z &= V_{z\,A_3}=0\\ V_z &= V_{z\,A_4}=0 \end{aligned} \tag{13}$$

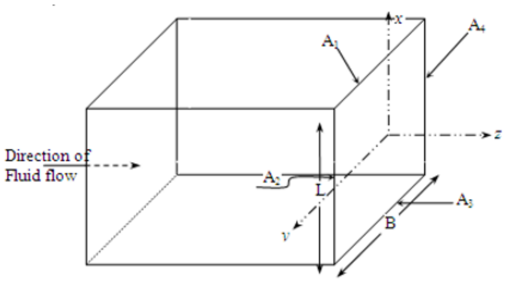

Fig. 7. Model for test case.

4. DISCRETIZATION

As shown in Fig. 8 in this meshfree method, the LWF is applied over a Weight Function.

A GWF of Eq. (11) can be obtained as:

$$\int_{\Omega_Q} W\left(\mu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}\right)-\frac{dP}{dz}-\rho\frac{\partial V_z}{\partial t}\right)d\Omega=0 \tag{14}$$

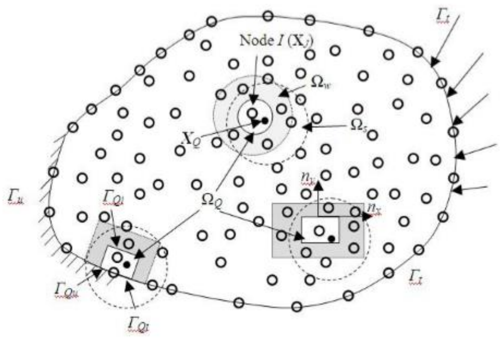

Fig. 8. Domain and boundaries.

Using the divergence theorem, the LWF is obtained as:

$$\int_{\Omega_Q}\left[\mu\left(\frac{\partial w}{\partial x}\frac{\partial V_z}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial V_z}{\partial y}\right)\right]d\Omega$$
$$-\int_{\Gamma_Q}\left[\mu\left(w\frac{\partial V_z}{\partial x}n_x+w\frac{\partial V_z}{\partial y}n_y\right)\right]d\Gamma \qquad (15)$$
$$+\int_{\Omega_Q}\rho w\frac{\partial V_z}{\partial t}d\Omega-\int_{\Omega_Q}w\,M\,d\Omega=0$$

Applying the initial boundary condition, Eq. (15), is obtained:

$$\int_{\Omega_Q}\left[\mu\left(\frac{\partial w}{\partial x}\frac{\partial V_z}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial V_z}{\partial y}\right)\right]d\Omega$$
$$-\int_{\Gamma_{Qi}}\left[\mu\left(w\frac{\partial V_z}{\partial x}n_x+w\frac{\partial V_z}{\partial y}n_y\right)\right]d\Gamma$$
$$-\int_{\Gamma_{Qu}}\left[\mu\left(w\frac{\partial V_z}{\partial x}n_x+w\frac{\partial V_z}{\partial y}n_y\right)\right]d\Gamma \qquad (16)$$
$$+\int_{\Omega_Q}\rho w\frac{\partial V_z}{\partial t}d\Omega$$
$$=\int_{\Omega_Q}w\,M\,d\Omega+\int_{\Gamma_{Qt}}\mu w\,\bar{q}\,d\Gamma$$

Quartic (4^{th} order) spline is applied as the weight function in this study as below:

$$W(\mathrm{x}-\mathrm{x}_I)=\hat{W}(\bar{d})=\begin{cases}1-6\bar{d}^2+8\bar{d}^3-3\bar{d}^4 & \bar{d}\le 1\\ 0 & \bar{d}>1\end{cases} \qquad (17)$$

where $\bar{d}=d_i/r_w$ is normalized distance.

Therefore in this research the quartic (4^{th} order) spline is applied as shown in Fig. 9:

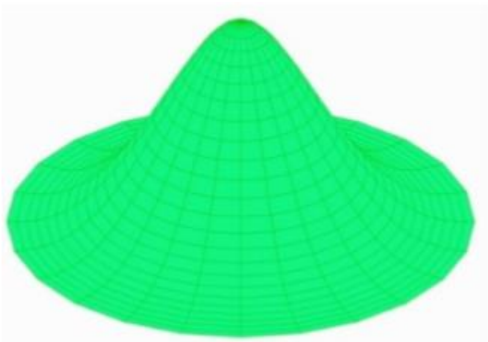

Fig. 9. Quartic (4th order) spline domain.

Table 1 Used parameter values in MLPG and FEM solutions

Parameter	Value	Unit
ρ	1000	kg/m^3
μ	2.5	Ns/m^2
L	0.10	m
B	0.10	m
$V_{z_{ini}}$	0	m/s
V_{z_A}	0	m/s
-∂P/∂z	4000	N/m^2/m
Δt	0.01	sec

The LWF, Eq. (16), resulted a relationship regarding to $\hat{V}_{z_i}$. Substitution of Eq. (7) into the Eq. (16) leads to the following:

$$[\mathrm{K}]\{V_z\}=\{\mathrm{f}\} \qquad (18)$$

where

$$K_{IJ}=\int_{\Omega_Q}\begin{bmatrix}\varphi_{I,y} & \varphi_{I,z}\end{bmatrix}\begin{bmatrix}\mu & 0\\ 0 & \mu\end{bmatrix}\begin{Bmatrix}\varphi_{J,y}\\ \varphi_{J,z}\end{Bmatrix}d\Omega \qquad (19))$$

$$f_I=\int_{\Omega_Q}M\,\varphi_I\,d\Omega \qquad (20)$$

5. COMPARISON WITH FEM

The flow field parameters of the fluid in SI units are as following:

Figure 10 shows the numerical results of the the unsteady incompressible fully developed fluid flow with advancing in time.

The comparison of velocity by the results of FEM with the present work under the same conditions at the location (x=0.025 m, y=0.025 m) of the duct cross section is presented in Table 2. Consequently the MLPG solution results are presented and compared with the results of the FEM solution in Fig. 10.

Figure 11 also presents the percent of deviation between MLPG and FEM methods in this research work.

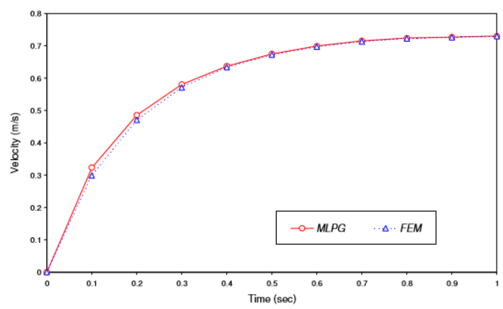

Fig. 10. Comparison of MLPG results with FEM.

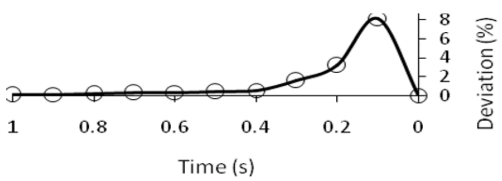

Fig. 11. Deviation between MLPG and FEM methods.

As can be seen from Fig. 11, the maximum deviation is about 8.09% which was occurred at the initial times and by passing the time the deviation between both methods goes to zero. The mean deviation is also about 1.32% for 1 s.

Table 2 Comparison of velocity by the results of FEM with the present work

Time (s)	MLPG Solution	FEM Solution
0.00	0.0000	0.0000
0.10	0.3234	0.2992
0.20	0.4853	0.4706
0.30	0.5802	0.5714
0.40	0.6371	0.6337
0.50	0.6747	0.6722
0.60	0.6992	0.6971
0.70	0.7151	0.7134
0.80	0.7237	0.7225
0.90	0.7268	0.7261
1.00	0.7298	0.7295

6. CONCLUSION

This article presents a meshfree method for the unsteady incompressible fully developed fluid flow. The trial function construction process is the most important part of the meshfree method implementation. In this article the RBFs are used for the process of the trial functions construction. A local weighted residual is extended by using a quartic (4^{th} order) spline as the interpolation function. In fact, this work most notably introduces two important innovations. A robust high-order MLPG meshfree based RBF method (RBF- MLPG) was developed to solve the 2- D unsteady incompressible fully developed fluid flows. In addition, a MLPG meshfree method with quartic (4^{th} order) spline as the weight function is presented for solving two-dimensional unsteady incompressible fully developed fluid flow. Finally, the numerical performance of the meshfree method is proved through comparison with the FEM outputs. This comparison shows that this meshfree method is very effective in producing good results.

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