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A Reliable Pressure Strain Correlation Model for Complex Turbulent Flows

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ABSTRACT

Developing an accurate and reliable model for the pressure strain correlation is a critical need for the success of the Reynolds Stress Modeling approach. This is challenging because replicating the non-local effects of pressure using a modeling basis composed of local tensors is limiting. In this paper we use physics based arguments and analysis of simulation data to select additional tensors to extend this modeling basis for pressure strain correlation modeling to formulate models with improved precision and robustness. We integrate these tensors in the modeling basis and develop separate models for the slow and rapid pressure strain correlation. This complete pressure strain correlation model is tested for different turbulent flows and its predictions are compared to prior pressure strain correlation models. We show that the new model with an extended tensor basis is able to show improvements in accuracy and reliability.

Keywords: Turbulence modeling; Computational fluid dynamics; Reynolds stress models; Pressure strain correlation.

NOMENCLATURE

b_{ij}	Reynolds stress anisotropy	u_i	fluctuating velocity component
d_{ij}	dissipation anisotropy tensor	U_i	mean velocity component
f_s	blending function		
k	turbulent kinetic energy	β	ellipticity parameter
L	length scale tensor	δ_{ij}	Kronecker delta tensor
p	fluctuating pressure	η_{ij}	dissipation tensor
P	mean pressure	ν	kinematic viscosity
P_{ij}	production tensor	ρ	density
R_{ij}	Reynolds stress tensor	ψ_{ij}	pressure strain correlation
$T_{ijk,k}$	diffusive transport term		

1. INTRODUCTION

Turbulent flows are important to problems in many fields of engineering and sciences like Mechanical Engineering, Chemical Engineering, Aerospace Engineering, Environmental Engineering, Oceanography, Civil Engineering, Meteorology, Astrophysics, etc. The ability to predict the evolution and properties of turbulent flows better will have impact in these fields. An example from Biomedical engineering turbulent flow of blood in the heart causes sclerosis (Wesolowski *et al*. 1965) leading to cardiovascular disorders like strokes and heart attacks (Al-Omari and Rousan, 2010). Improved understanding of turbulence can help Biomedical community to treat and cure such disorders. The number of people suffering from congestive heart failure (CHF) in the United States is estimated to be more than 4.5 million, and more than half a million additional people develop CHF each year. This situation is made worse by the low number of heart donors. Over the last five years, on an average less than 2,000 heart trans-plants were successfully carried out in the United States, a country with highly advanced medical facilities. Artificial heart devices like Ventricular Assist Devices (VAD) have emerged as the only alternative therapy for patients suffering from such heart diseases. Improved prediction of cardiovascular turbulence can aid in faster and more reliable design of these devices (Bachmann *et al*. 2000; Zhang *et al.* 2013).

In academic and industrial applications, most simulations into turbulent flow problems use turbulence models. Turbulence models are simplified relations relating high order quantities that are difficult to compute in terms of simpler flow parameters. These unknown correlations represent the actions of viscous dissipation, pressure-velocity interactions, etc. For example pressure strain correlation is a non-local phenomenon and is difficult to compute. Using models for pressure strain correlation, it is expressed as a function of Reynolds stresses, dissipation and mean velocity gradients which are local quantities. This enables estimation of the pressure strain correlation and its effects on flow evolution in a simpler manner which is computationally inexpensive also. Turbulence models are an important component of all computational fluid dynamics software and are used in almost all simulations of fluid flows of engineering importance.

Most industrial applications use simple two-equation turbulence models like the $k-\varepsilon$ and $k-\omega$ models. However recent emphasis in the scientific community has shifted to Reynolds stress models (Hanjalić and Launder 2011; Durbin 2017; Klifi and Lili 2013; Mishra and Girimaji 2014; Jakirlić and Maduta 2015; Manceau 2015; Eisfeld *et al*. 2016; Schwarzkopf *et al.* 2016; Moosaie and Manhart 2016; Lee *et al*. 2016; Mishra and Girimaji 2017; Panda *et al.* 2018; Sun *et al.* 2017; Mitra *et al.* 2019a,b). Reynolds stress models have the potential to provide better predictions than turbulent viscosity based models at a computational expense much lower than Direct Numerical Simulation (DNS) or Large Eddy Simulations (LES) based studies. They may be able to model the directional effects of the Reynolds stresses and additional complex interactions in turbulent flows (Johansson and Hallbäck 1994). They have the ability to accurately model the return to isotropy of decaying turbulence and the behavior of turbulence in the rapid distortion limit (Pope 2000). The important difference between the popular eddy viscosity models and Reynolds Stress Modeling method is the computation of the components of the Reynolds stress. In 2-equation models like the $k-\varepsilon$ and $k-\omega$ models, the turbulent kinetic energy is computed. Then the eddy-viscosity hypothesis is used to determine the components of the Reynolds stress tensor from the turbulent kinetic energy and the mean velocity gradient. The eddy viscosity hypothesis is somewhat accurate for simple sheared flows, it gets unsatisfactory for complex turbulent flows with mean streamline curvature, zones of recirculation, turbulent separation and reattachment, etc. Reynolds Stress Models use explicit transport equations for the components of the Reynolds stress. This ensures that these models can account for complex physics, deal with high degrees of anisotropy and provide more accurate results.

Reynolds stress models use equations for the transport of the individual components of the Reynolds stress tensor. This Reynolds stress transport equation forms the foundation of the Reynolds stress modeling approach and is given by Pope (2000).

$$\partial_t \overline{u_i u_j} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = P_{ij} - \frac{\partial T_{ijk}}{\partial x_k} - \eta_{ij} + \psi_{ij}$$

Where,

$$P_{ij} = -\overline{u_k u_j}\frac{\partial U_i}{\partial x_k} - \overline{u_i u_k}\frac{\partial U_j}{\partial x_k},$$

$$T_{kij} = \overline{u_i u_j u_k}\frac{\partial U_i}{\partial x_k} - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k} + \delta_{jk}\overline{u_i \frac{p}{\rho}} + \delta_{ik}\overline{u_j \frac{p}{\rho}}$$

$$\eta_{ij} = -2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}, \psi_{ij} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \quad (1)$$

The turbulence production process is represented by P_{ij} and represents transfer of energy from the mean velocity field to the turbulent fluctuations. η_{ij} represents the dissipation process where the turbulent kinetic energy is lost as internal energy. The turbulent transport process is represented by T_{ijk} and has contributions from viscous diffusion, pressure transport and turbulent convection. Finally ψ_{ij} represents the pressure strain correlation and redistributes turbulent kinetic energy among the components of the Reynolds stresses. Of these terms, production is the only process that is closed at the single point level. The other terms require models for their closure. The accuracy of the Reynolds stress modeling approach depends on the quality of the models developed for these turbulence processes. Out of these the modeling of the pressure strain correlation is considered to be the most important.

The pressure strain correlation of turbulence consists of two components- the slow pressure strain correlation modeling the non-linear interactions in between the fluctuating velocity field and the rapid pressure strain correlation modeling the interactions between the mean velocity and the fluctuating velocity field. This is shown in the Poisson equation for pressure Pope (2000).

$$\frac{1}{\rho}\nabla^2(p^R + p^S) = -2\frac{\partial U_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \quad (2)$$

Here p^R and p^S are the rapid and slow components of pressure. On the right-hand side of Eqs. (2), the first term represents linear interactions between the fluctuating velocity field with the mean velocity gradient and the second term represents the non-linear interactions in between the fluctuating velocity field.

Due to its importance there have been many attempts to develop closure models for the pressure strain correlation. Chou (1945) established the formulation for the second moment closure approach and introduced the pressure strain correlation term. Rotta (1951) developed a linear closure for the slow pressure strain correlation term using a modeling expression that was linear in the Reynolds stresses. Launder *et al.* (1975) developed a model for the complete pressure strain correlation. They developed a novel closure for the rapid pressure term and incorporated the model of Rotta (1951) for the slow pressure strain correlation. Jones and Musonge (1984) attempted to develop pressure strain correlation models that could be applicable for complex recirculating flows. Their model expression was similar to Launder *et al*. (1975) but the closure coefficients were calibrated to different values determined by the best agreement with their data for high Reynolds number homogeneous flows. Sarkar and Speziale (1990) developed a nonlinear extension for the slow pressurestrain correlation for high Reynolds number flows. This model was able to show improved agreement with the non-linear trends in the return to isotropy behavior. This was extended to a fully non-linear quadratic model for the complete pressure strain correlation in Speziale *et al*. (1991). Johansson and Hallbäck (1994) formulated a non-linear model for the rapid pressure strain correlation with terms in the expression that are of the fourth order in the Reynolds anisotropy tensor. This model showed improved agreement for some homogeneous turbulent flows.

There remain deficiencies in the performance of established models for the pressure strain correlation. These deficiencies are two-fold-limitations in accuracy and limitations in realizability.

Available pressure strain correlation models have unsatisfactory accuracy in many important classes of flows. For example in vorticity dominated flows their predictions may not be satisfactory. For these flows linear stability theory, experiments and DNS show growth in the turbulent kinetic energy. However established models predict that turbulence is decaying in these cases Blaisdell and Shariff (1996). Similarly the predictions of available pressure strain correlation models are inadequate in non-equilibrium turbulent flows, flows with swirl and re-circulation, etc. Mishra and Girimaji (2015, 2019).

Most pressure strain correlation models suffer from realizability violations. Realizability conditions ensure that the predictions of the turbulence model are consistent with a stochastic process. The pressure strain correlation models available lead to realizability violations at or in the neighborhood of the two-component limit of turbulence (Mishra and Girimaji, 2016). While the two-component limit is termed as a limiting state for the Reynolds stresses, it is found in many engineering flows. For example in near wall turbulence the state of the Reynolds stress tensor is extremely close to the two-component limit with the wall normal component of the Reynolds stresses being negligible. Such realizability violations in important flows limit the applicability of pressure strain correlation models.

Most classical pressure strain correlation models have focused on the closure modeling expression and the values of the closure coefficients to improve the performance of models. With respect to the model expression there has been a trend to-ward more complex terms that are non-linear in the Reynolds stress tensor Speziale *et al*. (1991). For example while the model of Launder *et al*. (1975) was linear in the Reynolds stress tensor, the model of Speziale *et al*. (1991) is quadratic and the model of Johansson and Hallbäck (1994) is fourth order. With respect to the closure coefficients, investigations have tried to calibrate them to more specialized data sets from experiments and DNS. Investigators have also made the closure coefficients functions of the invariants of the Reynolds stress tensor. This allows additional degrees of freedom in the modeling expression and enables better agreement with additional data sets. However the improvements due to such steps have been incremental. The central issues of unsatisfactory accuracy in specific important classes of flow or the issues with realizability are still present and important.

Some investigations have raised questions about the inadequacy of the modeling basis used in pressure strain correlation closures. The modeling basis is composed of the set of tensors used in the modeled constitutive equation for the pressure strain correlation. In classical one-point closure modeling these are one-point tensors including the Reynolds stress anisotropies, the turbulent kinetic energy and the dissipation. The set of tensors used in the modeling basis determines the type and extent of information about the turbulent flow field that is avail-able in the model formulation. In an incompressible flow pressure is governed by the Poisson equation. Due to the elliptic nature of this governing equation the pressure strain correlation is not a one-point tensor and attempts to model it using one-point tensors may be limited. Reynolds and Kassinos (1995) have claimed that in rotation-dominated turbulent flows, the modeling basis for the pressure strain correlation is limited. They introduced additional non-local tensors to the modeling basis like stropholysis, circulicity, etc. Cambon *et al*. (1992) have also claimed that additional tensors may be needed to model the pressure strain correlation in rotation-dominated flows. However both these models use non-local tensors that may not be avail-able in an engineering application. Mishra and Girimaji (2010) and Mishra and Girimaji (2013) have carried out a spectral analysis to outline the manner in which the modeling basis is limited and the manner in which it affects the ability of the model to replicate specific features of turbulent flows.

If the limitations in the pressure strain correlation models are due to limited modeling basis, there are three important questions to be answered:

1. What tensors need to be added to the modeling basis to have additional information that is relevant for modeling.
2. While many different correlations and turbulent statistics may be added to the modeling basis and may offer different degrees of benefit, we must identify the optimal tensors to be added.
3. Finally with these added tensors, how much improvement can we show in the performance of single-point pressure strain correlation models.

In this paper we address these questions in order. Using physics based arguments we outline a set of tensors to be added to the modeling basis for the slow pressure strain correlation and separately for the rapid pressure strain correlation. We show that these tensors add missing information to the modeling effort that is important to improve the potential performance of pressure strain correlation closures. We develop a complete model for the pressure strain correlation using this extended modeling basis. This model is tested for a range of mean flows while compared to DNS results. In this investigation, we use the popular models of Launder *et al*. (1975) and Speziale *et al*. (1991) for comparison. The present model shows improved agreement with DNS results and significant improvements over these earlier pressure strain correlation models.

2. THEORETICAL AND MATHEMATICAL DETAILS

In this section we outline our procedure to select specific tensors to the modeling basis for the pressure strain correlation. During this process, physical arguments for the choice of specific tensors and the particular benefits that they offer, with respect to the modeling of definite features of the pressure velocity interaction term. We demarcate this procedure sequentially, first for the slow pressure strain correlation model and then for the rapid pressure strain correlation model. During this selection, we try to consider tensors that are still single point and are available in the engineering single point modeling methodology. Following this selection, we develop the individual slow and rapid pressure strain correlation models with this expanded modeling basis.

2.1 Slow Pressure Strain Correlation Modeling Basis

Considering the slow pressure strain correlation model, we commence with the details of the rate of dissipation tensor. The rate of dissipation tensor can be decomposed into its deviatoric and isotropic components**:**

$$\eta_{ij} = D_{ij} + \frac{2}{3}\eta\delta_{ij} \tag{3}$$

Here, $\eta = \frac{\eta_{ii}}{2}$ and D_{ij} is the deviatoric component of the rate of dissipation tensor.

Traditionally, the deviatoric component of the rate of dissipation tensor is combined with the pressure strain correlation mechanism and the two are modeled together (Lumley and Newman, 1977)

$$\psi_{ij} = D_{ij} + \psi'_{ij} \tag{4}$$

In flows where the Reynolds number is large dissipation can be assumed to be isotropic $D_{ij} = 0$. In most Reynolds Stress Modeling investigations this assumption is adopted and it is assumed that the rate of dissipation tensor is nearly isotropic. For all practical modeling purposes, ψ_{ij} is the slow pressure strain correlation only. However recent direct numerical simulation studies suggest that this assumption is inadequate Kim *et al*. (1987); Lee and Reynolds (1985b). For example in near wall turbulence this assumption is unsatisfactory Lee and Reynolds (1985b). In fact Yeung and Brasseur (1991) have proved that if the large scale structures in a turbulent flow are anisotropic the small scale turbulent motions will have a significant level of anisotropy. Due to these arguments the assumption of the isotropy of the rate of dissipation is a significant shortcoming and causes deficiencies in the slow pressure strain correlation model. To address the shortcomings due to this assumption we introduce the dissipation anisotropy tensor (d_{ij}) in the modeling basis:

$$d_{ij} = \frac{\eta_{ij}}{\eta} - \frac{2}{3}\delta_{ij} \tag{5}$$

This tensor allows the model to have information about the anisotropy in the rate of dissipation mechanism and should improve the predictions of the models especially in the inhomogeneous flows.

There is a direct proportionality between anisotropy of large scale stress bearing eddies and the dissipative eddies at the small scales (Jakirlić and Hanjalić 2002). From this assumption, the dissipation anisotropy can be modeled in terms of the Reynolds stress anisotropy (Warrior *et al.* 2014).

$$d_{ij} = 2f_s b_{ij} \tag{6}$$

The blending function appeared in above equation ensures a smooth transition from anisotropic turbulence to isotropic turbulence (Jakirlić and Hanjalić, 2002). The blending function was defined in terms of the second and third invariants of Reynolds stress anisotropy as defined in (Warrior *et al.* 2014).

A considerable amount of information required for the closure modeling of the terms in the Reynolds Stress Models is contained in the two-point correlation tensor, $R_{ij}(\vec{x}, \vec{r}) = u_i(\vec{x})u_j(\vec{x}+\vec{r})$. The two-point correlation contains significant information about the dissipation and the pressure strain correlation, both of which can be expressed as functionals of the two-point correlation. The two-point correlation also has important information about the turbulent length scales. As the two-point correlation is non-local it is not used in the single-point modeling basis. This causes another significant shortcoming in classical Reynolds Stress Models that is the assumption of a single integral length scale. This is markedly true in flows where the geometry of the flow domain or body forces lead to a coordinate direction in the flow being decidedly preferred. For example axisymmetric expansion and axisymmetric contraction mean flows. In many anisotropic turbulent flows, the characteristic length scale is observed to be varying in different directions Panda *et al*. (2017); Tietjens and Prandtl (1934). At the most basic level, we must try to include this anisotropy in the length scale in the modeling basis for the pressure strain correlation. We introduce the length scale anisotropy tensor (l_{ij}) in the modeling basis and derive it as follows. The length scale information tensor (L_{ij}) is defined as:

$$\eta_{ij} = \frac{3}{4}\frac{k^{\frac{3}{2}}}{\eta}(c^*{}_1 b_{ij} + c^*{}_2 d_{ij}) \tag{7}$$

A scaling factor l is defined as $\frac{k^{3/2}}{\eta}$. The expression for the length scale anisotropy is given by

$$l_{ij} = \frac{L_{ij}}{l} = \frac{3}{4}(c^{*}{}_1 b_{ij} + c^{*}{}_2 d_{ij}) \quad (8)$$

This derivation can be found in detail in Panda *et al*. (2017).

2.2 Rapid Pressure Strain Correlation Modeling Basis

Considering the rapid (or linear) pressure strain correlation term, we address the level of information used to characterize the state of the turbulent flow field. One of the key shortcomings in the Reynolds Stress Modeling approach to pressure strain correlation closures is the use of only the Reynolds stress tensor to describe the state of the turbulent flow field. This leads to a coarse grained description that limits the potential accuracy of the rapid pressure strain correlation model. Mishra (2014); Mishra and Girimaji (2010, 2013, 2014, 2015) have made important insights about the specific limitations due to this level of characterization of the turbulent flow field. They have shown that turbulent flow fields with the same Reynolds stresses can have very different internal structuring and lead to very different evolution Mishra and Girimaji (2010, 2013). Using spectral analysis, they have established a universal evolution (termed the statistically most likely behavior) that is dependent on the mean velocity gradient. This behavior was shown to be highly dependent on the mean velocity gradient of the flow Mishra and Girimaji (2013, 2014). At the primary level, including information about the local mean velocity gradients may be a good substitute for detailed multi-point information about the internal structuring of the turbulent flow field. For information about the mean velocity gradient, we introduce the invariants of the mean velocity gradient in the modeling basis. In this paper we restrict ourselves to planar mean velocity gradients. Here the information about the mean velocity field can be included using the ellipticity parameter Mishra (2014); Panda and Warrior (2018):

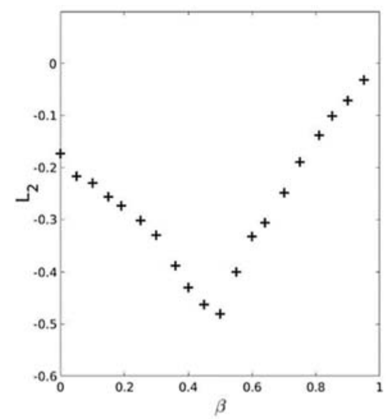
$$\beta = \frac{\mathrm{W_{mn}+W_{mn}}}{\mathrm{W_{mn}W_{mn}+S_{mn}S_{mn}}} \quad (9)$$

2.3 Integration of Additional Tensors Into Model Expressions

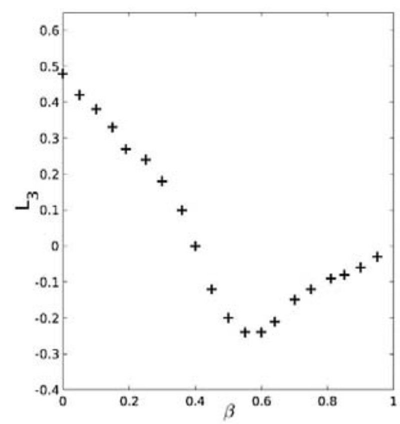
To integrate these three tensors into the model expression we adopt a practical recourse. For the slow pressure strain correlation, the addition of the tensors requires that the model expression be extended. On experimenting with variants where the coefficients of the closure expression were made functions of d_{ij} and l_{ij}, we found the final model to not perform well. However the general expression for the rapid pressure strain correlation closure is retained and the closure coefficients are made functions of these three tensors. On experimenting with variations (where additional terms involving these tensors were included in the model expression) we have found that this does not negatively affect the performance of the new model. Additionally we hope that retaining the established closure expression and only changing the nature of the closure coefficients will encourage the scientific community to incorporate this model into their proprietary codes.

A most general form of the slow pressure strain correlation can be written as:

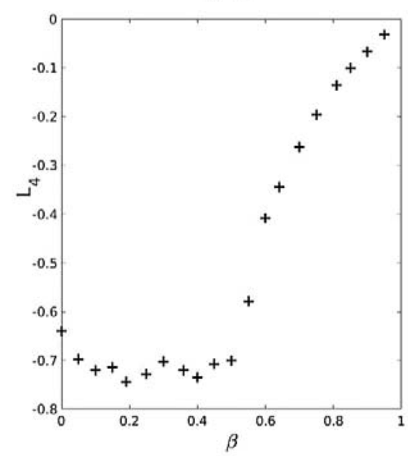
$$\psi_{ij}^{(s)} = \beta_1 b_{ij} + \beta_2 (b_{ik} b_{kj} - \tfrac{1}{3}\mathrm{II}_b \delta_{ij}) \quad (10)$$



(a) L_2



(b) L_3



(c) L_4

Fig. 1. Calculated values of the rapid pressure strain correlation model coefficients L_2,L_3 and L_4 as functions of β (Panda and Warrior, 2018).

Here b_{ij} is the Reynolds stress anisotropy, II_b is the second invariant of the Reynolds stress anisotropy tensor. In the most general case, β_1 and β_2 are assumed to be functions of the second and third invariants of the Reynolds stress anisotropy tensor. The model of Rotta (1951) assumed $\beta 1$ to be a constant and β_2 to be zero. Sarkar and Speziale (1990) assumed both β_1 and β_2 to be non-zero constants. The slow pressure-strain correlation used in this investigation has the model expression derived by Panda *et al.* (2017). This involves the most general expression for the slow pressure strain correlation in terms of the three tensors b_{ij},d_{ij} and l_{ij}.

$$\psi_{ij}^{(s)} = c_1 b_{ij} + c_2 d_{ij} + c_3 l_{ij} + c_4(b_{ik}b_{kj} - \tfrac{1}{3}b_{mn}b_{mn}\delta_{ij})$$
$$+c_5\left(b_{ik}b_{kj} - \tfrac{1}{3}b_{mn}d_{mn}\delta_{ij}\right) + c_6(d_{ik}b_{kj} - \tfrac{1}{3}d_{mn}d_{mn}\delta_{ij})$$
$$+c_7\left(b_{ik}l_{kj} - \tfrac{1}{3}b_{mn}l_{mn}\delta_{ij}\right) + c_8(d_{ik}l_{kj} - \tfrac{1}{3}d_{mn}l_{mn}\delta_{ij})$$
$$+c_9\left(l_{ik}b_{kj} - \tfrac{1}{3}l_{mn}l_{mn}\delta_{ij}\right). \qquad (11)$$

The values of the model coefficients have been derived in Panda *et al.* (2017) and are given by (*c1,c2,c3,c4,c5,c6,c7,c8,c9*)=(3.1,1.1,−0.6,−4.3,−15 .8,−7.2,8.4,6.6,9.8).

Considering the rapid pressure strain correlation, the linear form of the model expression is

$$\frac{\psi_{ij}^R}{k} = C_2 S_{ij} + C_3(b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}b_{mn}S_{mn}\delta_{ij}) + C_4(b_{ik}W_{jk} + b_{jk}W_{ik}) \qquad (12)$$

Here $b_{ij} = \frac{\overline{u_i u_j}}{2k} - \frac{\delta_{ij}}{3}$ is the Reynolds stress anisotropy tensor, S_{ij} is the rate of strain term for the mean velocity field and W_{ij} is the rate of rotation term for the mean velocity field. Following the notation of Speziale *et al.* (1991) C_2,C_3 and C_4 are the coefficients of the rapid pressure strain correlation model.

Based on this form of the rapid pressure strain correlation, the Reynolds stress anisotropy evolution equation is derived from the Reynolds stress transport equation at the rapid distortion limit

$$\frac{db_{ij}}{dt} = 2b_{ij}b_{mn}S_{mn} + L_2 S_{ij} + L_3(b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}b_{mn}S_{mn}\delta_{ij}) + L_4(b_{ik}W_{jk} + b_{jk}W_{ik}) \qquad (13)$$

Here $L_2 = \frac{C_2}{2} - \frac{2}{3}$, $L_3 = \frac{C_3}{2} - 1$ and $L_4 = \frac{C_4}{2} - 1$. Once the form of the rapid pressure strain correlation model expression is fixed, the modeling reduces to determine the values of the model coefficients C_2,C_3 and C_4 (or of L_2,L_3 and L_4). To integrate the additional tensor in the rapid pressure expression (the ellipticity parameter, carrying information about the mean velocity gradient) we follow the outline of Mishra and Girimaji (2013, 2014). Here the model coefficients are made explicit functions of the ellipticity parameter. To this end, we use representation theory and try to ensure that the stationary state of the anisotropy evolution equation (Eq. (12)) matches the stationary state of the Reynolds stress anisotropy tensor observed in RDT simulations Devaney (2008). Using representation theory the values of the Reynolds stress anisotropy at equilibrium can be expressed as a polynomial function in terms of the mean rate of strain and mean rate of rotation. Based on Pope Pope (1975); Panda and Warrior (2018), the general form of this is given by

$$b_{ij} = G_1 S_{ij} + G_2(S_{ik}W_{kj} + W_{ki}S_{kj}) + G_3(S_{ik}S_{kj} + \tfrac{(\beta-1)\delta_{ij}}{3}) \qquad (14)$$

G_1,G_2 and G_3 are scalars that are functions of the invariants of flow statistics. This approach can be extended to three dimensional mean flow cases (Mishra and Girimaji (2015); Panda and Warrior (2018)). In this paper, we study two dimensional mean flow cases that can be completely described using β. Using the polynomial form from Eq. (13), and using a Matlab script to calculate values of the Reynolds stress anisotropies at the stationary equilibrium points (designated by b^*_{11}, b^*_{22} and b^*_{12}), G_1,G_2 and G_3 can be expressed in terms of these stationary values of the Reynolds stress anisotropies

$$G_1 = \frac{b_{11}^* - b_{22}^*}{\sqrt{2(1-\beta)}}$$

$$G_2 = -\frac{b_{12}^*}{\sqrt{\beta(1-\beta)}}$$

$$G_1 = \frac{b_{11}^* + b_{22}^*}{(1-\beta)/3} \qquad (15)$$

At the stationary states for the Reynolds stress anisotropy, the evolution equation Eq. (12) simplifies to

$$2b_{ij}^* b_{mn}^* S_{mn} + L_2 S_{ij} + L_3 (b_{ik}^* S_{jk} + b_{jk}^* S_{ik} - \tfrac{2}{3} b_{mn}^* S_{mn}\delta_{ij}) + L_4(b_{ik}^* W_{jk} + b_{jk}^* W_{ik}) = 0 \qquad (16)$$

Here b_{ij}^* is the value of the Reynolds stress anisotropy at the stationary state.

Replacing b^*_{ij} in Eq. (15) by the polynomial form from Eq. (13), we get a equation for the coefficients L_2,L_3,L_4 as functions of G_1,G_2,G_3

$$L_2 = -2(1-\beta)G_1^2 - 4\beta(1-\beta)G_2^2 + \frac{(1-\beta)^2}{3}G_3^2$$

$$L_3 = -(1-\beta)G_3$$

$$L_4 = 2(1-\beta)G_2 \qquad (17)$$

According to Crow (1968) the value of the C_2 (and the L_2) coefficient should be fixed. This Crow Constraint just improves the initial evolution of the turbulent flow. In this paper we choose to not follow this constraint. Many popular models like the model of Speziale *et al.* (1991) do not follow this constraint. Substituting the values of G_1,G_2 and G_3 computed in Eq. (14) into the Eq. (16), we get the values of L_2,L_3 and L4. The detailed steps of derivation is available in Panda and Warrior (2018).

At this point, we have outlined the additional tensors to be added to the modeling basis for the pressure strain correlation and the specific reasons for their addition. The slow and rapid pressure strain correlation models with these additional tensors have been formulated. In the next section we use these two model expressions together to simulate the evolution of general turbulent flows that are far off the limiting states of turbulence. This methodology follows the procedure counseled by Speziale *et al*. (1992), where they have warned against testing models of the rapid and slow pressure strain correlation in isolation. During the test cases, the turbulent kinetic energy ($k = \frac{\overline{u_i u_i}}{2}$) evolves as

$$\frac{dk}{dt} = P - \eta \tag{18}$$

The modeled evolution equation for the dissipation is

$$\frac{d\eta}{dt} = C_{\eta 1}\frac{\eta}{k}P - C_{\eta 2}\frac{\eta^2}{k} \tag{19}$$

The values of the coefficients are taken as $C_{\eta 1} = 1.44$ and $C_{\eta 2} = 1.88$.

3. RESULTS AND DISCUSSION

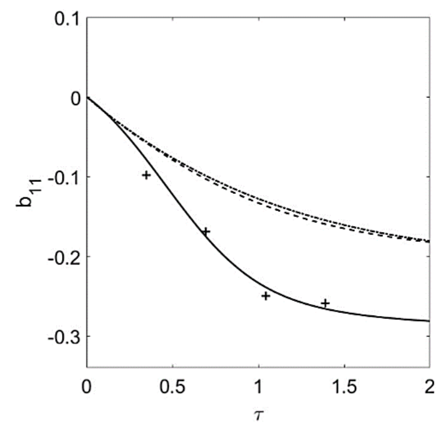
In this section, the present pressure strain correlation model is tested for a wide variety of general turbulent flows. We ensure that these flows are general in the sense that they are not at the limiting states of decaying turbulence or the rapid distortion limit. We use the predictions of established models by Launder *et al*. (1975) and Speziale *et al*. (1991) as yard sticks to compare the performance of the present model.

The case of a homogeneous turbulent flow under mean plane strain is a benchmark test case for turbulence models. This situation happens when the fluid flow is acted upon only by load forces that are parallel to them. The fluid particle is stretched along one directions and squeezed along a perpendicular axis. In Fig. 2 the evolution of Reynolds stress anisotropy and turbulent kinetic energy is shown for plane strain mean flow. The present model predictions are shown in a solid line, the SSG and the LRR model are shown in dash-dot and dotted lines respectively. DNS data from Lee and Reynolds (1985a) is shown using unfilled circles in the figure. The predictions of the present model for both the components of the Reynolds stress anisotropy and the evolution of the turbulent kinetic energy show agreement with the DNS data. The present model is able to show some improvement in comparison to the predictions of popular models like those by Launder *et al*. (1975) and Speziale *et al*. (1991).

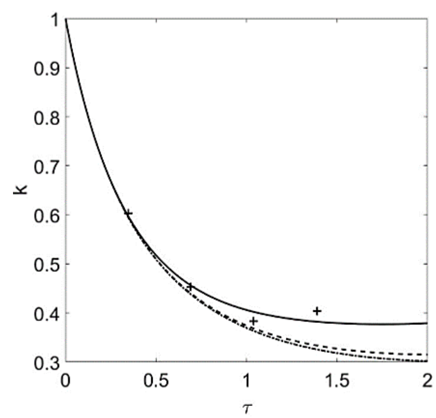
Turbulent flows that are dominated by rotation effects are common in many aerospace applications. Mean rotation dominated flows include many flows of importance in aerospace applications such as trailing vortex (Govindaraju and Saffman, 1971), flap-edge vortex (Brooks and Marcolini, 1986), leading-edge vortex flows, etc. This limitation means that these models (and by extension, the en-tire Reynolds Stress Modeling approach) may not be suitable for application in such important problems.

It is documented that the LRR and SSG models may not give satisfactory performance in many elliptic streamline flows. Blaisdell and Shariff (1996) have simulated homogeneous turbulence subjected to elliptic mean flows:

$$\frac{\partial U_i}{\partial x_j} = \begin{bmatrix} 0 & 0 & -\gamma - e \\ 0 & 0 & 0 \\ \gamma - e & 0 & 0 \end{bmatrix} \tag{20}$$



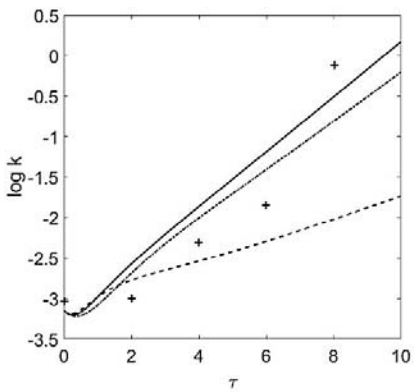
(a) b_{11}



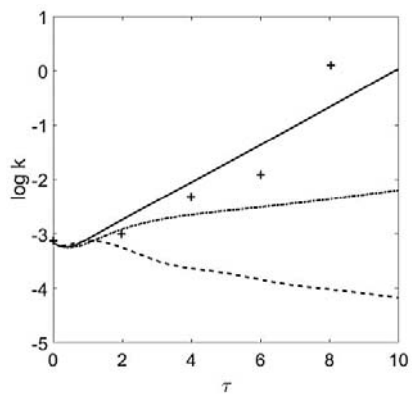
(b) k

Fig. 2. Evolution of a) the Reynolds stress anisotropy b_{11} b) turbulent kinetic energy k for plane strain mean flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Lee and Reynolds Lee and Reynolds (1985a) is included for comparison.

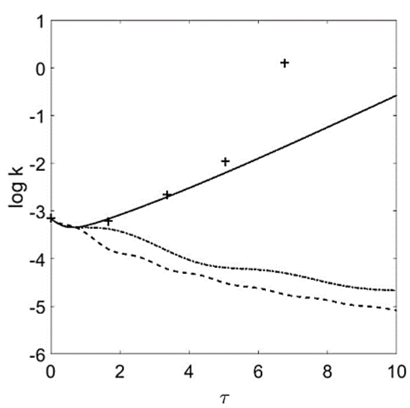
where $e = \sqrt{\frac{1-\beta}{2}}$ and $\gamma = \sqrt{\frac{\beta}{2}}$. For $e > \gamma$ the mean flow has elliptic streamlines of aspect ratio $E = \sqrt{(\gamma + e)(\gamma - e)}$. We use this data from 3 simulations with mean flows having aspect ratios E=3,2 and 1.5. The turbulent velocity field is initially isotropic and the initial $\frac{\eta}{Sk} = 0.167$.



(a) $k, E = 1.5$



(b) $k, E = 2$



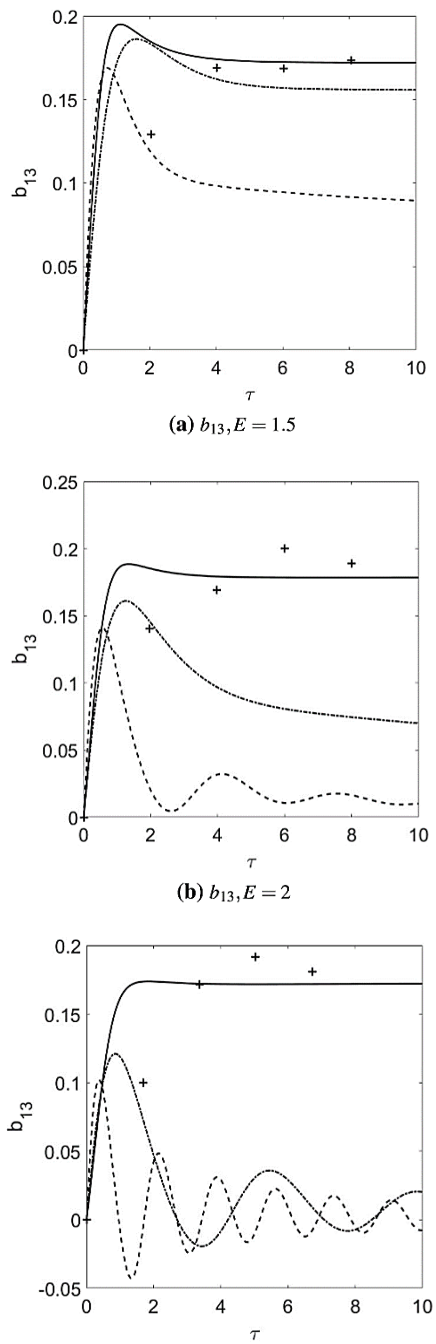
(c) $k, E = 3$

Fig. 3. Turbulent kinetic energy evolution for elliptic flows a) E=1.5 b) E=2 c) E=3. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff Blaisdell and Shariff (1996) is included for comparison.

In Figs. 3 and 4 we use the data from the direct numerical simulations of Blaisdell and Shariff (1996) in elliptic streamline mean flows. Figure 3 represents the time evolution of turbulence kinetic energy for elliptic mean flow with different values of aspect ratio. For case E=1.5 in Fig. 3 (a) the LLR and SSG models predict turbulent kinetic energy growth but at a rate much lower than the DNS of Blaisdell and Shariff (1996). As the relative strength of mean rotation effect increases, in Figure 3 (b) and (c), the performance of LLR and SSG becomes less satisfactory. For the case E=3 the LLR and SSG models predict turbulent kinetic energy decay but the DNS predicts turbulent kinetic energy growth. For all 3 cases the predictions of the present model are in agreement with the DNS data qualitatively and quantitatively. Unlike LRR and SSG models, the present model predicts growth of turbulent kinetic energy for all three cases of elliptic streamline mean flow. The rate of growth of turbulent kinetic energy predicted by the present model is able to show quantitative agreement with the DNS data also.

In Fig. 4, the evolution of Reynolds stress anisotropy (b_{13} component) is shown. For all three values of aspect ratio the new model predictions shows improvement agreement with the DNS data of Blaisdell and Shariff (1996). Testing across a variety of elliptic streamline flows seems to suggest that the present model is able to show significant improvement in predictions of both the Reynolds stress anisotropy and the turbulent kinetic energy evolution.

An important feature to test in pressure strain correlation models is the interaction and the relative accuracy of the slow and rapid pressure strain correlation models. This needs to be tested in isolation and also in tandem at different values of Sk/ε (Mishra, 2010; Mishra *et al.* 2016). The hurdle for such testing is to find a reliable set of data from DNS or experimental studies where the only parameter that varies is Sk/ε over a large margin. In Figs. 5 and 6, we perform a exhaustive validation for the case of homogeneous sheared mean flow. This flow case is of great importance theoretically and from the point of view of the engineering applicability of the model. We use the data from Isaza and Collins (2009) where the evolution of the Reynolds stress anisotropies and the turbulent kinetic energy was collected for a range of different shear parameter S^*. This is important as it tests the performance of the slow and rapid pressure strain correlation models when used in conjunction with each other. This issue is emphasized in Speziale *et al*. (1992) where the authors comment that testing the rapid and slow pressure strain correlation models in isolation can lead to unsound and misleading results. Testing the complete pressure strain correlation model, for a range of in this manner acts as an exhaustive test of entire pressure strain correlation model as a unit where the rapid and slow models work in conjunction with each other. We select three specific cases of the shear parameter from Isaza and Collins (2009), a) $S^*= 3$ b) $S^*= 15$ c) $S^*= 27$. At $S^*= 3$, the nonlinear behavior is dominant in the flow physics and the performance depends more on the accuracy of slow pressure strain model. At $S^* = 27$, the linear behavior is dominant and the performance depends more on the ac-curacy of rapid pressure strain model. Finally, At $S^*= 15$, both linear and non-linear physics are of equal importance in the turbulence evolution. This case tests how well the entire pressure strain correlation model works as a unit. The present model predictions matches well with the DNS data for all three values of the shear parameter. There is a significant improvement over the predictions of both the LLR and SSG models.

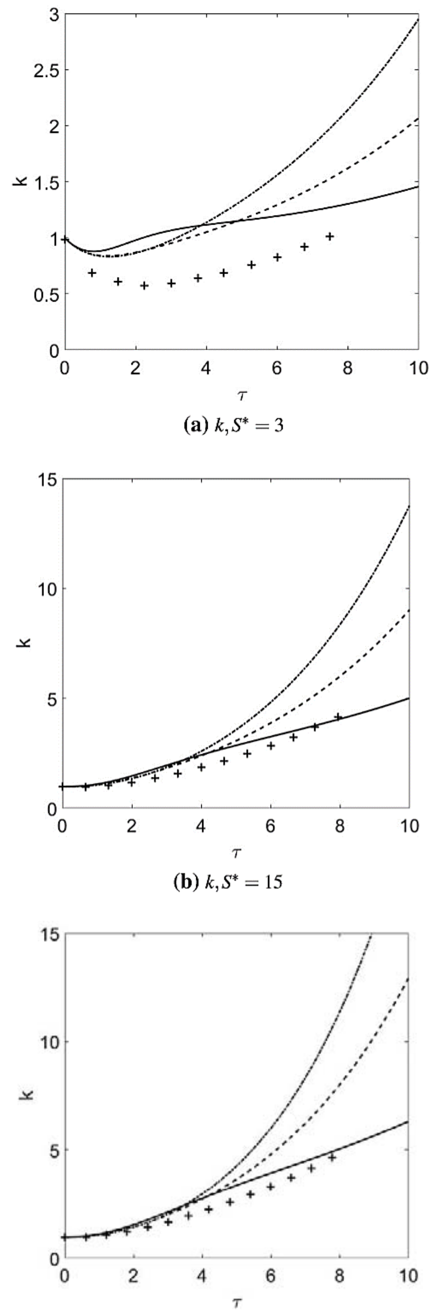


(a) $b_{13}, E = 1.5$

(b) $b_{13}, E = 2$

(c) $b_{13}, E = 3$

Fig. 4. Reynolds stress anisotropy b_{13} evolution for elliptic flows a) E=1.5 b) E=2 c) E=3. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff Blaisdell and Shariff (1996) is included for comparison.



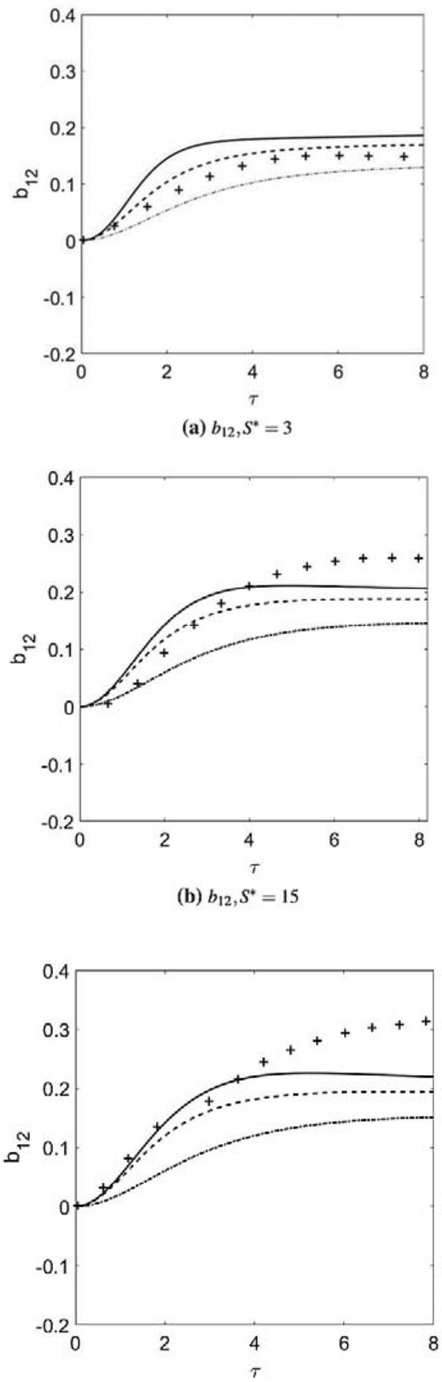
(a) $k, S^{*} = 3$

(b) $k, S^{*} = 15$

(c) $k, S^{*} = 27$

Fig. 5. Turbulence kinetic evolution for purely sheared flows a) S^{*}=3 b) S^{*}=15 c) S^{*}=27. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins Isaza and Collins (2009) is included for comparison.

In Fig. 7, the present model prediction of the evolution of turbulence kinetic energy is compared with the large eddy simulation data of Bardina *et al.* (1983) for purely sheared flows. The predictions of the present model are in reasonable agreement with the LES data and show accuracy at par with the models of LLR and SSG.

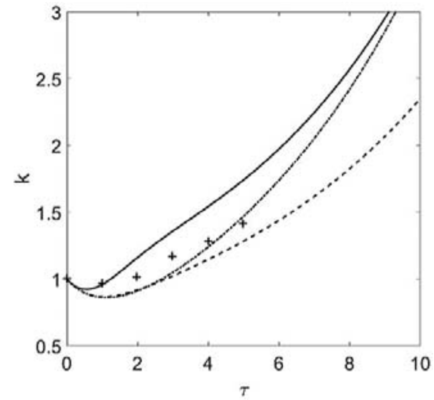


(a) $b_{12}, S^* = 3$

(b) $b_{12}, S^* = 15$

(c) $b_{12}, S^* = 27$

Fig. 6. Reynolds stress anisotropy b12 for purely sheared flows a) S^*=3 b) S^*=15 c) S^*=27. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins Isaza and Collins (2009) is included for comparison.



(a) $b_{12}, S^* = 3$

Fig. 7. Evolution of turbulent kinetic energy for the purely sheared flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Bardina *et al.* Bardina *et al.* (1983) is included for comparison.

In testing across these flows we find that the present model is able to show some improvements in accuracy for strain dominated flows like multiple examples of homogeneous shear flow Bardina *et al.* (1983); Isaza and Collins (2009) and plane strain flow Lee and Reynolds (1985a). For rotation dominated flows like those investigated by Blaisdell and Shariff (1996) the present model shows much improvement over the established models of SSG Speziale *et al.* (1991) and LRR Launder *et al.* (1975).

4. CONCLUSIONS

It is accepted in the turbulence modeling community that the pressure strain correlation model is a critical component for the success of the Reynolds Stress Modeling approach. Pressure strain correlation models try to capture the effects of the interaction of fluctuating pressure with the fluctuating rate of strain tensor. Such models try to express the effects of the pressure strain correlation using a tensor basis of local tensors like Reynolds stresses, disspation and mean velocity gradients. The physics that the pressure strain correlation model tries to capture is non-local due to the non-local nature of pressure. Using a limited set of local tensors to capture this physics leads to limitations in model performance. In this investigation we extend the tensor basis used for pressure strain correlation modeling. This set of additional tensors sequentially justified based on physics based arguments. We formulate a model using this extended modeling basis. The present model is tested for a wide variety of turbulent flows and contrasted against the predictions of other popular models like those by Launder *et al.* (1975) and Speziale *et al.* (1991). It is shown that the new model provides significant improvement in predictive accuracy. We are currently testing this pressure strain correlation model for inhomogeneous turbulent flows where the effects of boundaries and walls are important. This article aims to communicate the promising performance of this model in homogeneous turbulent flows to the turbulence modeling community.

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