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On the Stabilization of a Viscoelastic Jeffreys Fluid Layer Heated from Below

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Abstract

Feedback control is applied to the problem of a viscoelastic Jeffreys fluid layer heated from below to investigate conditions for delay of the onset of convection. Interesting results for fixed Prandtl number 1 and 10 were found showing that for some conditions proportional control may not work as expected. Also, some limits of the feedback control in terms of the parameters of the system through an analytical approach by mean of the Galerkin method are discussed. In order to complete the study a numerical analysis was also performed to map the space of physical parameters. The results of this work are discussed and compared with results of previous authors while attention to small control adjustments is paid.

Keywords: Feedback control; Hydrodynamic stability; Rayleigh convection.

Nomenclature

E	dimensionless retardation time	$\mathbf{u}$	dimensionless fluid velocity
F	dimensionless relaxation time	W	vertical velocity perturbation
g	acceleration due to gravity		
H	fluid layer depth	β	thermal expansion coefficient
k	perturbation wavenumber	γ	controller gain
$k_{x,y}$ x, y	wavenumber projection	Θ	temperature perturbation
K	controller gain	κ	thermal diffusivity
Pr	Prandtl number	λ	stress relaxation time
Ra	Rayleigh number	ν	fluid kinematic viscosity
ΔT^*	temperature difference across the fluid layer	μ	fluid dynamic viscosity
T	dimensionless temperature	σ	complex parameter
T_B^*	bottom wall temperature	σ_R	perturbations growth rate
T_T^*	top wall temperature	ω	frequency of oscillation

1. Introduction

The problem of hydrodynamic instability in viscoelastic fluid layers heated from below, which have been investigated since several decades ago, is considered here in the light of a feedback control strategy suggested by the work of Tang and Bau (1993a) and Tang and Bau (1993b) for prevention of convective motions in newtonian fluids. However, convection in viscoelastic Jeffreys fluid layers sets in as oscillatory motions questioning if proportional control would also stabilize the fluid as in the newtonian case. At the same time further coupling of parameters of the system under this feedback control strategy are expected, raising other questions on the physics of the problem and limitations of this approach to control of convection.

The present manuscript is devoted to study the effect of a feedback proportional control strategy on the problem of Rayleigh convection in viscoelastic Jeffreys and Maxwell fluids. Previous works gave an interesting perspective for the present study since not only different hydrodynamical problems have been considered but mechanisms for control as well. To the best knowledge of the authors two early reports introducing theoretical pr $+\mathbf{u}\cdot\nabla T=\nabla^2 T$ oposals for feedback control of thermal convection are due to Singer *et al.* (1991) and to Singer and Bau (1991), who also performed an experimental study, shown to be effective and trigger further studies. Wang *et al.* (1992) also studied theoretically and experimentally the feedback control on thermal convection and state that this strategy may induce chaotic motion in time dependent convection which is important in the light of the results presented in this manuscript. Next, in a series of papers by Tang and Bau (1993b), by Tang and Bau (1994), by Tang and Bau (1995) and by Tang and Bau (1998) the stabilization of a newtonian fluid layer heated from below is achieved with a feedback control strategy based on the thermal boundary condition at the bottom wall and on temperature measurements in the fluid layer. The stabilization of Rayleigh convection in a saturated porous media, subject to feedback control based on direct temperature measurements, was numerically studied by Tang and Bau (1993a) who found that for certain situations proportional control fails. Also, Howle (1997a) and Howle (1997b) introduced the idea of real time temperature measurements, through the shadowgraph technique, to the proportional feedback control loop which proof to be successful in the case of Rayleigh convection too.

On the other hand, applications based on pattern formation in polymeric liquids (see the works of Mitov and Kumacheva (1998) and Li *et al.* (2000), por example) may be related to studies mentioned above and, to some extent, to the nonlinear, hydrodynamic stability analysis in layers of newtonian fluids, reported by Shortis and Hall (1996). In their work Shortis and Hall (1996) found that linear and nonlinear controllers may delay the onset of convection and hexagonal patterns are sustained for certain control conditions. The conclusions of Shortis and Hall (1996) suggest that attention to the problem convection in viscoelastic fluids could be useful for applications, as the ones mentioned previously. Thus, it is reasonable to think that application of proportional feedback control to delay the onset of convection in viscoelastic Jeffreys fluid layers are relevant.

Then, in order to investigate how the critical Rayleigh number, wavenumber and frequency of oscillation are affected by the proportional control, analytical and numerical calculations shall be performed by the known Galerkin technique (Finlayson 1972). Since oscillations in the fluid are expected it would of interest to investigate the relationship between the parameters of the system and that of the controller.

The manuscript is organized as follows. Section 2. is devoted to the formulation of the problem and statement of feedback control strategy is presented. Analytical and numerical calculations are presented in the Linear Stability Analysis section 3. Results and discussion on the findings are shown in section 4. Finally, conclusions are exposed in section 5.

2. MATHEMATICAL FORMULATION

Here, effect of proportional control on the hydrodynamic stability of a viscoelastic Jeffreys fluid layer, of infinite horizontal extent, uniformly heated from below and cooled from above is considered. As the viscoelastic fluid is heated the onset of convective motions is expected, across the layer depth H, so that the proportional feedback control strategy used by Tang and Bau (1993a) can be used. The governing equations are those for mass conservation, for momentum balance, for heat conduction and the constitutive equation for the viscoelastic Jeffreys fluid. After some simplifications these model equations can be written in dimensionless form as

$$L_1\left[\Pr^{-1}\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla P-\mathrm{Ra}T\mathbf{k}\right] \quad (1)$$
$$=L_2\nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\nabla^2 T \quad (2)$$

where L_1 and L_2 are linear operators defined as

$$L_1=1+F\frac{\partial}{\partial t} \quad (3)$$

$$L_2=1+EF\frac{\partial}{\partial t} \quad (4)$$

and $\mathrm{Ra}=g\beta\Delta T^{*}H^3/\nu\kappa$, $\Pr=\nu/\kappa$, $F=\lambda_1\kappa/H^2$ and $E=\lambda_2/\lambda_1$ .Notice that the constitutive equation for the viscoelastic Jeffreys fluid is already being coupled to the momentum balance Eq. (1). For the uncontrolled hydrodynamic problem the governing Eqs. (1-2) were subjected to boundary conditions corresponding to rigid solid walls and perfect thermal conducting walls

$$\mathbf{u}=0,\quad \textbf{at}\quad z=\pm\frac{1}{2}T \quad (5)$$

$$T=0,\quad \textbf{at}\quad z=\pm\frac{1}{2} \quad (6)$$

However, as a proportional control strategy shall be implemented for suppression of the onset of convection a modification to the thermal boundary condition Eq. (6) at the bottom is considered. The thermal boundary at the bottom is then rewritten as

$$T=F_c\left(T_0\right),\quad \textbf{at}\quad z=-\frac{1}{2} \quad (7)$$

where F_c is a function defining the controllers response depending on the temperature measured in the middle of the layer, $z=0$. In view of the good results obtained by Tang and Bau (1993a), by Tang and Bau (1994) and by Tang and Bau (1995), for the stabilization of the fluid in the case of Rayleigh convection, the same idea is embraced here but with little observations from the physical point of view. For the present study attention to the effect of small feedback corrections via the controller is given. The idea of small adjustments comes from the fact that perturbations to the fluid are small to, and this shall be discussed later. Thus, for the case of proportional control Eq.(7) is expressed as

$$T=-KT_0, \quad \textbf{at} \quad z=-\frac{1}{2} \tag{8}$$

where $T_0=T(z=0)$ and K is interpreted as the response of the controller to a deviation from a desired temperature value at $z=0$ and known as the proportional control gain. This is the main idea behind the applied control approach.

3. LINEAR STABILITY ANALYSIS

The hydrodynamic stability of the fluid layer shall be subjected to a theoretical feedback control loop in order to avoid the onset of convective motions. In other words, the basic state of the system is subjected to perturbations that may lead to thermal convection but proportional control is introduced as a counterweight to the effect of perturbations in the system. Therefore, what follows is the perturbation of the governing Eqs. (1-2) according to

$$\mathbf{u}=\grave{o}\mathbf{u}_1 \tag{9}$$

$$T=T_0+\grave{o}T_1 \tag{10}$$

$$P=P_H+\grave{o}P_1 \tag{11}$$

where T_0 and P_H account for the basic state already included in the governing Eqs. (1-2) and the subscript 0 indicates the perturbation variables. Next, substitution of Eqs. (11) into system of Eqs. (1-2) produce the perturbed governing equations whose stability shall be studied. On the other hand, pressure still coupled to fluid velocity and temperature, which can be split after operating twice $\nabla\times$ on the perturbed momentum balance equation. The resulting perturbed governing equations are

$$L_1 Pr^{-1}\frac{\partial\nabla^2 w_1}{\partial t}=L_2\nabla^4 w_1+RaL_1\nabla^2 T_1 \tag{12}$$

$$\frac{\partial T_1}{\partial t}-w_1=\nabla^2 T_1 \tag{13}$$

where Eq. (12) is the momentum vertical component since this is independent from the horizontal ones. Notice, that subscript 1 in Eqs. (12-13) comes from the linear perturbation given in Eq. (11). Furthermore, periodic patters are expected as convective motions set in across the fluid layer, either as steady or oscillatory motions. Then, it is reasonable to seek for solutions in terms of normal modes as follow

$$\begin{aligned} w_1&=W(z)\exp\left[i\left(k_x x+k_y y\right)+\sigma t\right]\\ T_1&=\theta(z)\exp\left[i\left(k_x x+k_y y\right)+\sigma t\right]\end{aligned} \tag{14}$$

which allows the model for the hydrodynamic stability, Eqs. (12-13), is then reduced to the following ordinary differential equation system

$$\begin{aligned}&(1+F\sigma)\left[Pr^{-1}\sigma\left(D^2-k^2\right)W+Rak^2\right]\\&=(1+EF\sigma)\left(D^2-k^2\right)^2 W\end{aligned} \tag{15}$$

$$\left[\sigma-\left(D^2-k^2\right)\right]\theta=W \tag{16}$$

subject to the following, mechanical and thermal, boundary conditions

$$W=0 \quad \textbf{at} \quad z=\pm\frac{1}{2} \tag{16}$$

$$\theta=-KT_0 \quad \textbf{at} \quad z=\pm\frac{1}{2} \tag{17}$$

For the above equations and through the manuscript $D=d/dz$. In the following subsections, the Galerkin technique (Finlayson 1972) shall be used to treat the eigenvalue problem for the Rayleigh number \$Ra\$ presented by Eqs. (15-17). This is a convenient tool for the present type of problem allowing to critical conditions for the onset of convection without solving the differential equations.

3.1 Analytical Approach

In this section a Galerkin low order approximation is developed to investigate the relationship among the physical parameters of the system and the gain of the controller K. Besides, these analytical investigation may help to find limitations of the feedback control strategy for the problem in hand. Then, the following trial functions satisfying the boundary conditions Eqs. (17) are chosen

$$W_n=A_1(2z-1)^2(2z-1)^2+\ldots \tag{18}$$

$$\theta_n=B_1\left[z^2-\frac{Kz+1}{2(K+2)}\right]+\ldots \tag{19}$$

Next, after substitution of Eqs. (19) into the perturbed governing Eqs. (15-16) the residual is formed by multiplying each governing equation by its corresponding trial function and integrating across the fluid layer. In this way, a solvability condition, from which Ra is obtained, can be formed. This is

$$\begin{vmatrix}(W_m,L_3W_n) & (W_m,L_4\theta_n)\\ (\theta_m,W_n) & (\theta_m,L_5\theta_n)\end{vmatrix}=0 \tag{20}$$

where the linear operators L_1, L_2 and L_3 are defined as

$$\begin{aligned} &L_3=\left(1+F\sigma\right)Pr^{-1}\sigma\left(D^2-k^2\right)\\ &-\left(1+EF\sigma\right)\left(D^2-k^2\right)^2\\ &L_4=\left(1+F\sigma\right)Rak^2\\ &L_5=\left(D^2-k^2\right)-\sigma \end{aligned} \tag{21}$$

and $(,)$ indicates integrals across the fluid layer. A first approximation with $n=m=1$ produce an analytical expression for Pr, F, E, σ, Ra, k and K that is not accurate but keeps the tendency of the curves of criticality for the fluid layer hydrodynamics. From the point of view of the authors, these relationships are of interest for deeper understanding of the effect of K and its limitation on the stabilization of the layer. Since oscillatory convection is the instability mechanism for viscoelastic Jeffreys fluid layers heated from below the frequency of oscillation ω is introduced through $\sigma=\sigma_R+i\omega$.

At the first approximation the determinant in Eq. (20) outputs a linear complex expression for Ra, so that this parameter can be easily isolated. On the other hand, since Ra is real the frequency of oscillation is such that $Im(Ra)$ vanishes. In order to investigate the effect of the controller gain K on the hydrodynamics of the fluid a brief asymptotic analysis for $K<<1$ shall be considered. A physical explanation for this decision comes from the fact that small perturbations would require small K corrections too. Thus, Ra and ω can be expanded as

$$\begin{aligned} &Ra=Ra_0+Ra_1K+O\left(K^2\right)\\ &\omega=\omega_0+\omega_1K+O\left(K^2\right),\quad \textbf{for}\quad K<<1 \end{aligned} \tag{22}$$

For expansions Eqs. (22) it was confirmed that Ra_0 and ω_0 are in agreement with results reported by Takashima (1972) and by Sokolov and Tanner (1972) when $K=0$, as expected. For example, for viscoelastic Jeffreys fluids with $(Pr,F,E)=(1,0.1,0.1)$ it was found that $(Ra_{0c},k_c,\omega_{0c})=(1219.3,4.36,10.93)$ which gives a maximum error 2.55\%, 0.68\% and 1.05\%, correspondingly for each parameter, in comparison with the results of Takashima (1972).

$$Ra_1=\frac{7}{81}\frac{\left[\left(EFf_1+1\right)f_0^2EPr^2+\left[\left(2Ff_2+1\right)f_1E+k^2+70\right]f_3f_0Pr+\left(Ff_1+1\right)f_2^2f_3^2\right]\left(f_0EFPr+f_3\right)}{\left[\left(EPr+1\right)f_0-2k^2-384\right]^2F^2k^2Pr} \tag{23}$$

$$\omega_1^2=-\frac{5\left[\left(k^4+24k^2+504\right)EFPr+k^2+12\right]\left(E-1\right)\left(k^4+24k^2+504\right)Pr}{F^2\left[\left(k^4+24k^2+504\right)EPr+\left(k^2+10\right)\left(k^2+12\right)\right]^2} \tag{24}$$

For this approximation neither Ra_1 nor ω_1, shown in Eqs. (23-24), are independent of any of the properties of the fluid meaning that the gain control influence the whole system. Ra_1 contribution is given in Eq. (23) where the frequency of oscillation have already been introduced. At this point, some features of Ra_1 and ω_1 to understand the role of K are of interest. The Prandtl number Pr and the wavenumber k seem to be the parameters through which the control gain works due to the powers and change in sign of the factors these are involved in.

Where

$$\begin{aligned} &f_0=k^4+24k^2+504\\ &f_1=k^2-50\\ &f_2=k^2+10\\ &f_3=6k^2+12 \end{aligned} \tag{25}$$

Since the Prandtl number may have an important role on the feedback control strategy, a short calculation was made and for Ra_1 small controller gain corrections $(K<<1)$ are magnified in fluids with $Pr<<1$. For this case the Rayleigh number Ra_1 can be approximated, for $K,Pr<<1$, as

$$Ra_1\approx-\frac{7}{81}\frac{\left[f_1F+1\right]f_2^2f_3^3}{f_0^2PrF^2k^2}+\ldots \tag{26}$$

It is noticeable that in this limit, Ra_1 does not depend on the dimensionless retardation time E possibly due to a large thermal diffusivity. A further calculation of the corresponding critical wavenumber gives

$$k_c^2=19-\frac{1}{2F} \tag{27}$$

which is valid only for non zero viscoelastic effects. Another physical limitation for Eq. (27) is that k_c must be positive so that the relaxation time F can not be too small. Furthermore, it can be seen that the size of k_c may determine the assessment or not of the controller gain to the stabilization of the fluid due to the change of sign possibility in f_1 (see Eq.

25).

In the limit of fluids with $\mathrm{Pr} >> 1$ small controller gains corrections $\left(\mathrm{K} << 1\right)$ seem to have little effect on the Rayleigh number Ra_1 which is reduced to

$$\mathrm{Ra}_1 \approx -\frac{7}{81}\frac{\left[\mathrm{f}_1\mathrm{EF}+1\right]\mathrm{f}_0}{\mathrm{Fk}^2}+\ldots \tag{28}$$

However, in this limit the retardation time E contributes as a fluid stabilization factor and also sign changes may occur for certain values of $\mathrm{k_c}$. For this case, the critical wavenumber $\mathrm{k_c}$ can also be easily found to be

$$\mathrm{k}_\mathrm{c}^2 = \frac{26\mathrm{EF}-1}{3\mathrm{EF}} \pm \frac{\sqrt{4\left(691\mathrm{EF}-31\right)\mathrm{EF}+1}}{3\mathrm{EF}} \tag{29}$$

Notice that Eqs. (28-29) may hold only for non zero viscoelastic effects since non physical results could be obtained. In this case, the role of viscoelasticity is more important since it defines the critical wavenumber and later if the control gain correction assists or not the stabilization of the fluid. A common found behavior is the stabilization benefits of the proportional control strategy for the system although there are certain limits due to the viscoelasticity or to the viscoelastic fluid model itself, perhaps.

3.2 Numerical Computations

The intricate relationship among the parameters of the problem justifies a more general view of the onset of convection. A more accurate investigation of the role played by the proportional control strategy is now performed through a numerical analysis based on a variation of the Galerkin method previously used. Since governing Eq. (15) can be solved for temperature once W is known then the eigenvalue problem can be approached with very good convergence at a low order approximation. Thus, with W given by expansion Eq. (19) and the temperature being determined from Eq. (15) subject to the thermal boundary conditions Eqs. (17) a new solvability condition can be obtained after making orthogonal the velocity equation to its corresponding trial function. This condition is

$$\left|\left(\mathrm{W_m}, \mathrm{L_3W_n}\right)+\left(\mathrm{W_m}, \mathrm{L_4}\theta_\mathrm{n}\right)\right| = 0 \tag{30}$$

The process followed, starting in Eq. (30), to compute the critical conditions at which convection sets in is very similar to that in subsection 3.1. For the present investigation it has been considered that small corrections through the gain of proportional control strategy are sufficient to make the fluid layer more stable or delay the onset of convection. As previously stated in the analytical investigation, the small perturbations introduced earlier may not need large responses to stabilize the system. Then, the proportional controller gain shall be mapped from 0, for the uncontrolled problem, to 1.

For the numerical computations the viscoelastic fluid properties were fixed to $\mathrm{F} = 0.1, 1, 10$ and $\mathrm{E} = 0.05, 0.1$ while the Prandtl number to $\mathrm{Pr} = 1, 10$ to map the space of parameters involved. For these choices of properties a set of curves of criticality for the Rayleigh number, wavenumber and frequency of oscillation are to be calculated. All numerical results presented in the manuscript were obtained from the solvability condition Eq. (30) at third order of approximation.

3.2.1 Numerical Validation

Checks on numerical calculations were made for the case of classical Rayleigh convection, for Newtonian fluids, based on the results of Chandrasekhar (1981) and of Tang and Bau (1993a). For this case, when $K = 0$ the critical conditions obtained $\left(Ra_c, k_c\right) = \left(1707.76, 3.13\right)$, with the program built for this investigation, are in very good agreement with reported previously by Chandrasekhar (1981). Very good convergence of the numerical calculations was assured at the third order approximation as it is shown in Fig. 1, in the case of, Newtonian fluid, Rayleigh convection. the curve of criticality, at third order of approximation, was compared to that of Tang and Bau (1993a) and very good agreement was found too. Also, for the range of K from 0 to 1 a comparison between the second and third order approximations shows a maximum error smaller than 1%.

Further checks on the numerical calculations related to the case of viscoelastic Maxwell and viscoelastic Jeffreys fluids were carried out too. For example, in the case of viscoelastic Maxwell fluids with $K = 0$, $E = 0$ and $F = 0.1$ it was found $\left(Ra_c, k_c, \omega_c\right) = \left(870.55, 4.93, 15.09\right)$ which is in very good agreement with reports of (Pérez-Reyes and Dávalos-Orozco 2011) and (Takashima 1972), for example.

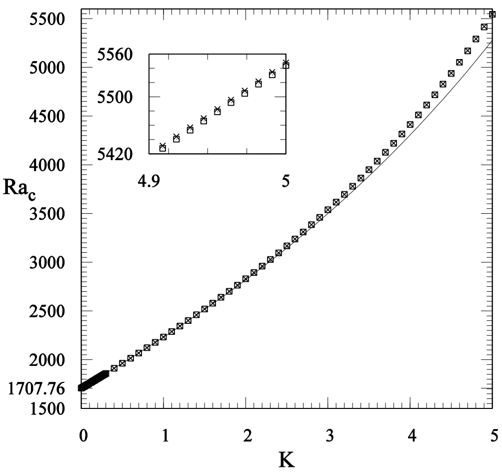


Fig. 1. Plots of Ra_c against K showing the convergence of the numerical calculations from the first order approximation, represented by the solid line, to the third approximation, represented by the dashed line. Notice, in the magnification, that the second order approximation, represented by the dotted line, and the third approximation are very close.

4. RESULTS AND DISCUSSION

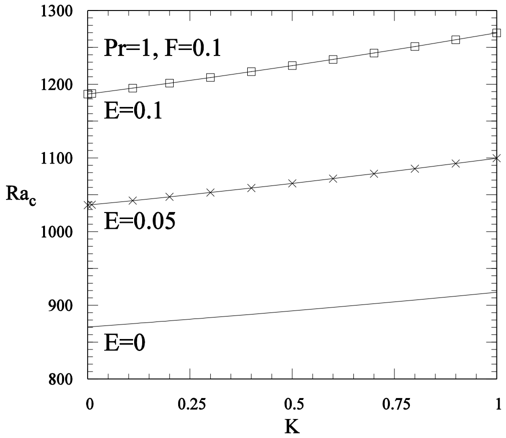
The results obtained from the numerical computations are presented for two viscoelastic fluid models, that of Maxwell and that of Jeffreys. Advantage of Prandtl number Pr fixed to 1 and 10 shall be taken to present and discuss the curves of criticality.

4.1 The Role of the Prandtl Number Pr

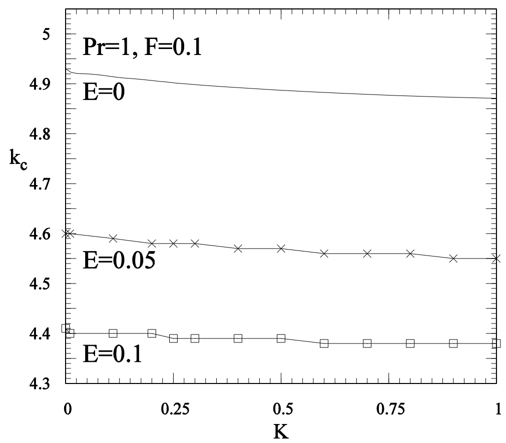
For the case of $Pr = 1$ and viscoelastic Maxwell fluids $E = 0$ occurs that the critical Rayleigh number Ra_c increases with the control gain K which means that the fluid layer is stabilized as shown in Figs. 2a,3a,4a. However, the effect of the stabilization is observed to decrease from fluids with $F = 0.1$ to fluids with $F = 10$. The curves for the critical wavenumber k_c have a change from fluids with $F = 1$, see Fig. 2b, to fluids with $F = 1,10$, see Figs. 3b,4b , and there is possibly a relaxation time at which the number of convective rolls stop increasing and start decreasing in size. Although the values of k_c are of the same order for all relaxation times, larger dependencies on K appear in Maxwell fluids $F = 1,10$. The critical frequency of oscillation ω_c always increases with K for the three values of the relaxation time, as shown in Figs. 2c,3c,4c, and this is reasonable since this could be a mechanism for energy dissipation.

For viscoelastic Jeffreys fluids with $Pr = 1$ the hydrodynamics is very similar to that of Maxwell fluids. However as it can be observed in Figs. 2a,3a,4a the effect of the control gain K on the system is more remarkable, as the retardation time E is increased, in comparison with all viscoelastic Maxwell fluids considered. The behavior of the critical wavenumber k_c, see Figs. 2b,3b,4b, and critical frequency of oscillation ω_c, see Figs. 2c,3c,4c, is very similar, in change rates with K, to the case of viscoelastic Maxwell fluids with differences in magnitudes only. In fact, results of k_c and ω_c for retardation time $E = 0.05, 0.1$ are always very close in magnitude as the relaxation time is increased.

The case of viscoelastic fluids with $Pr = 100$ seems to be interesting since for Maxwell fluids with relaxation time $F = 0.1,1$, see Figs. 5a,6a , the system becomes less stable with the control gain K ; but Maxwell fluids with relaxation time $F = 10$ behave different becoming more stable with K. This behavior points to the existence of a critical relaxation point at which the system stop being destabilized by K and starts being stabilized. For all relaxation times, the critical wavenumber always decreases for the used range of K as shown in Figs. 5b, 6b,7b. Notice, in Fig. 7b, that the wavenumber tends to become independent of K for relaxation time $F = 10$. On the other hand, the critical frequency of oscillation ω_c, see Figs. 5c, 6c, 7c, always decreases with K which may be explained if high thermal dissipation is considered through the fluid oscillations. This is, as K increases from zero the number of rolls decreases delaying heat transport across the fluid layer and allowing a more efficient thermal dissipation by smaller oscillations.



(a)



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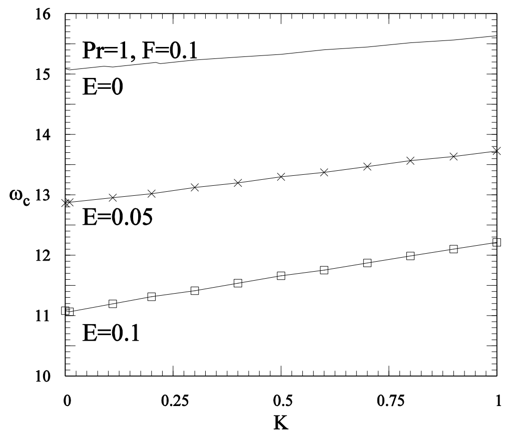
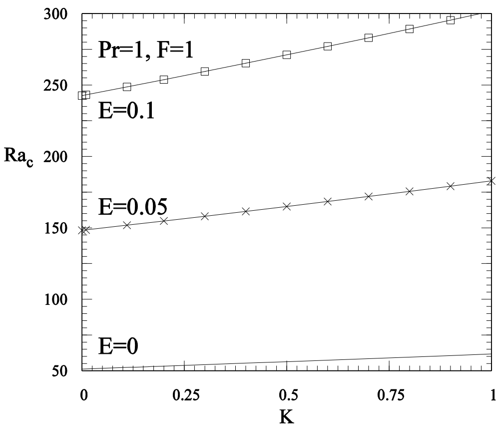
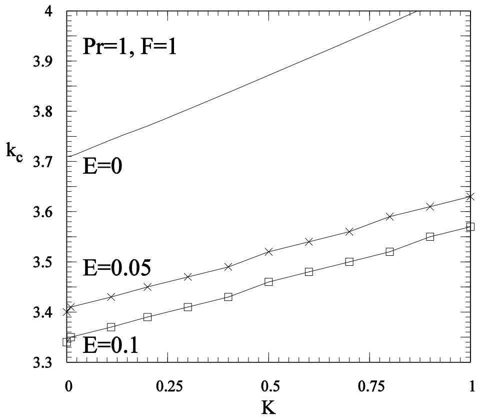


(c)

Fig. 2. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 2a, for k_c against K in Fig. 2b and for ω_c against K in Fig. 2c.



(a)



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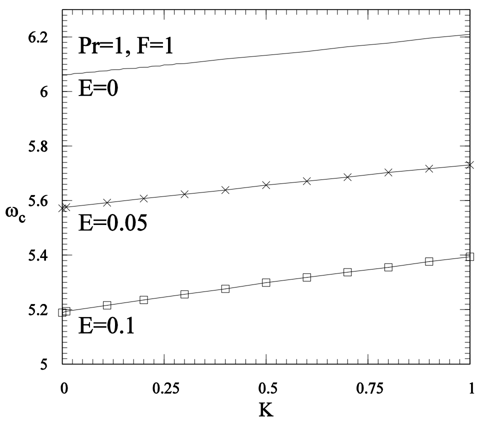
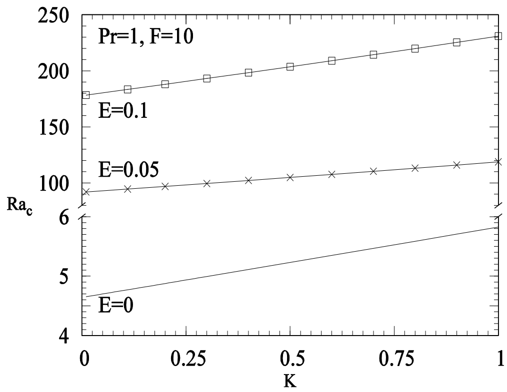


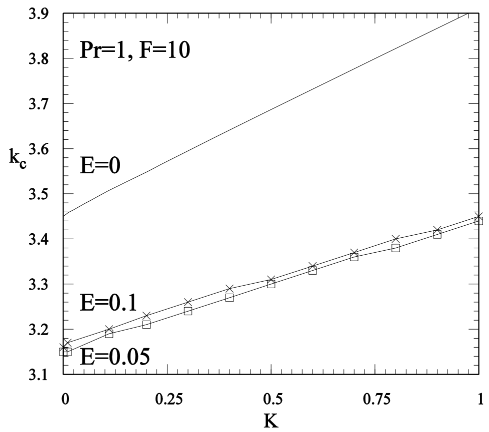
(c)

Fig. 3. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 3a, for k_c against K in Fig. 3b and for ω_c against K in Fig. 3c.

For viscoelastic Jeffreys fluids with $Pr = 100$ the system behaves different. For short, the control gain has a stabilizing effect which is magnified as the retardation time increases as shown in Figs. 5a,6a,7a. In fact, it seems that there is a coupling between E and F for this result since as F is increased the stabilization effect increases too. The critical wavenumber k_c , see Figs. 5b,6b,7b, and the critical frequency of oscillation ω_c , see Figs. 5c,6c,7c, behave in a very similar way. However, the heat transfer mechanism changes to an opposite direction from that arising in Maxwell fluids since ω_c tend to grow as K is increased. A physical explanation for this is as follows. The fluid becomes more stable by increasing the number of rolls with the relaxation time which is a more efficient heat transport configuration triggering oscillations in the fluid as well.



(a)



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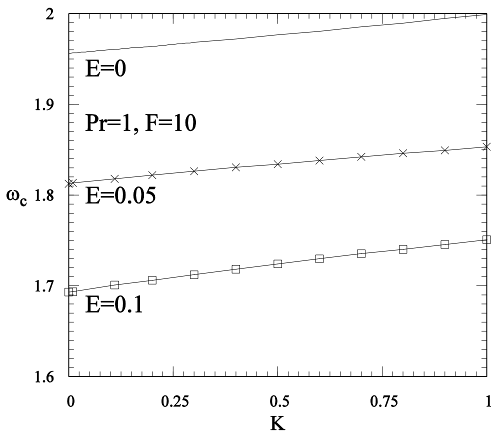
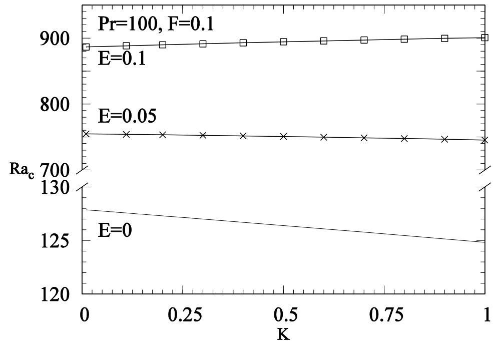
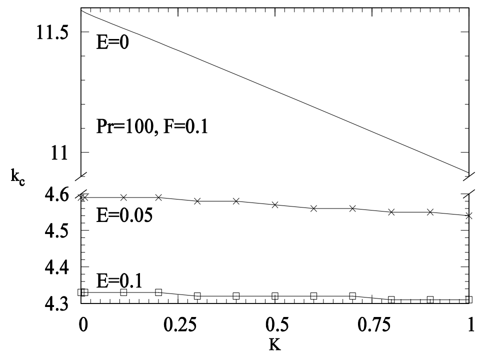


(c)

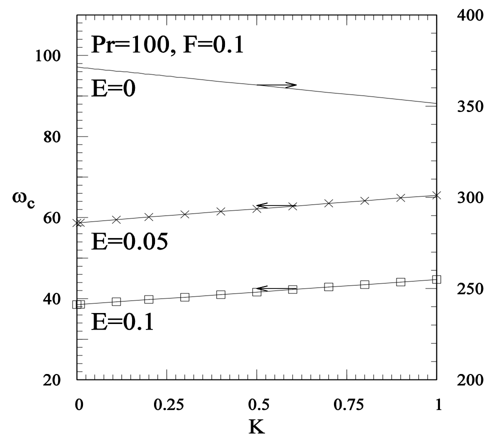
Fig. 4. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 4a, for k_c against K in Fig. 4b and for ω_c against K in Fig. 4c.



(a)



(b)



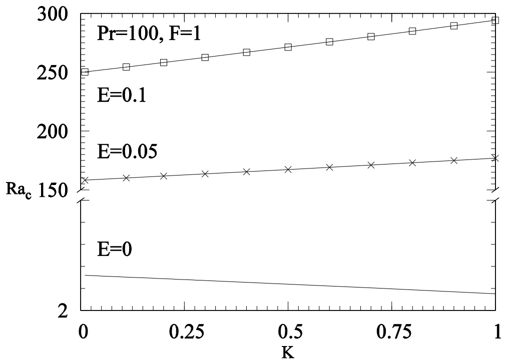
(c)

Fig. 5. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 5a, for k_c against K in Fig. 5b and for ω_c against K in Fig. 5c.

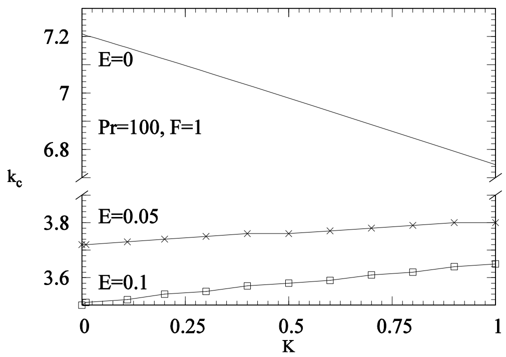
In some curves a nonlinear dependency of Ra_c , k_c and ω_c with K appears for some cases. This an interesting matter discussed later in section 4.3.

4.2 Viscoelastic Fluid Response

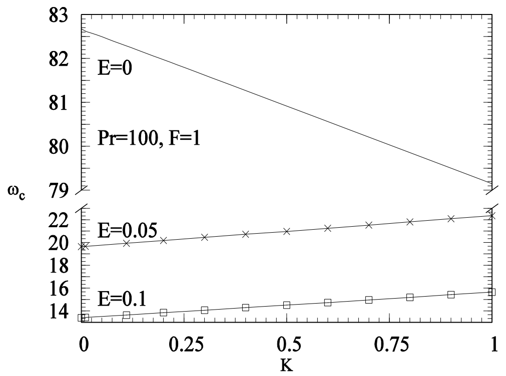
An interesting result is that the control proportional gain *K* has put on display the different responses of viscoelastic Maxwell and Jeffreys fluids under the stabilization strategy. For most of the cases, for viscoelastic Jeffreys fluids, the feedback proportional control was successful as is shown in Figs. 2-7. However, Fig. 5a shows an exception where Ra_c slowly decreases with *K*. This is an unexpected result in view of a comparison with the Rayleigh convection in Newtonian fluids.



(a)



(b)



(c)

Fig. 6. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 6a, for k_c against K in Fig. 6b and for ω_c against K in Fig. 6c.

Time dependent viscoelastic fluid motion is indeed a very complex subject where coupling of relaxation and retardation time can be observed through the precious curves of criticality. In other words, for some cases onset of convection is delayed in viscoelastic Maxwell and Jeffreys fluids as shown for example in Figs. 4a; for some cases the onset of convection is delayed only for one the viscoelastic fluids as shown for example in Figs. 6a; and for some cases as shown for example in Fig. 5a the response to the proportional control gain *K* may change with the retardation time.

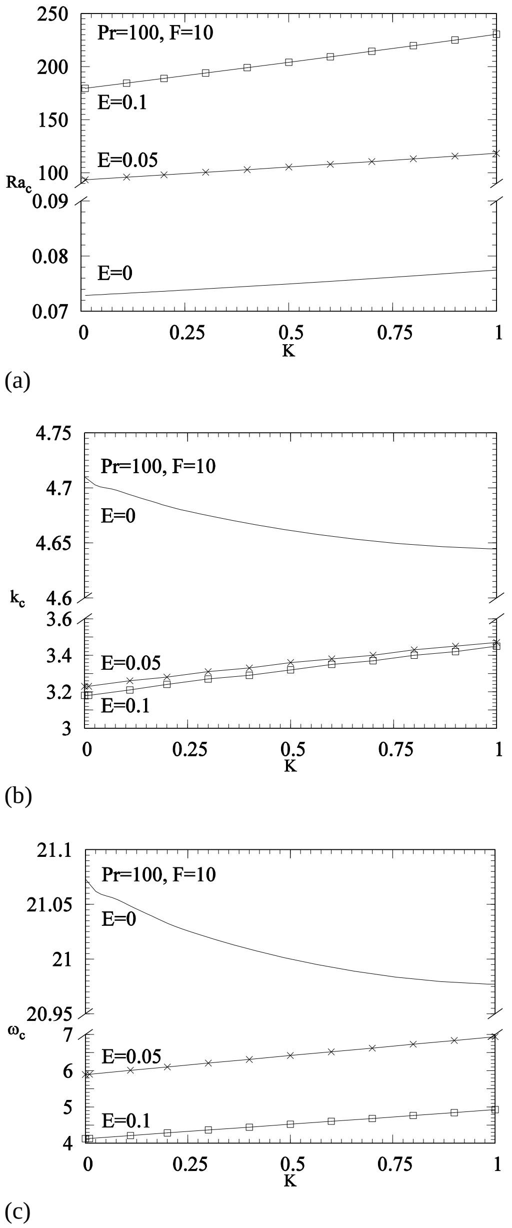


Fig. 7. Curves of criticality in viscoelastic Maxwell and viscoelastic Jeffreys fluids for Ra_c against K in Fig. 7a, for k_c against K in Fig. 7b and for ω_c against K in Fig. 7c.

4.3 Effect of Proportional Control Gain

From the curves of criticality presented before a main finding is that the proportional control gain K is not always successful for stabilizing or delaying the onset of convection in the fluid layer as expected in view of results for the classical problem of Rayleigh convection reported by Tang and Bau (1993a). The analytical approach to this problem to investigate the effect of small corrections through the feedback control strategy shows that retardation and relaxation times favor the stabilization of the layer or play against it.

In view of the nonlinear dependency shown by some curves of criticality, as in Fig. 7 for example, further numerical computations for K =1−10 were made looking for a more general view of the hydrodynamics. It was found that nonlinearity of those curves increase for larger values of K. For example, in the case of Pr = 1, F = 0.1 and E = 0.05,0.1 the curves for k_c have the shape of a parabola with a minimum between K = 1 and 1.5.

4.4 Perspective from Experimental Results

Despite the interesting results, to the best knowledge of the authors, there are no reports on the control of Rayleigh convection in viscoelastic fluid layers but some findings in the development of lab applications (see the work of Braun (2004) for example) point out that further understanding and optimization of the convective flow are needed. Although linear stability is not a concern in the work of Braun (2004) the importance of achieving the convection critical conditions and the control of heat transport across the fluid is evident. On the other hand, the results of Braun (2004) have also motivated an ongoing experimental study of the linear stability analysis of viscoelastic fluid layers.

5. CONCLUSION

In this paper the effect of proportional feedback control on the natural convection in a viscoelastic Jeffreys fluid layer was studied. An analytical and a numerical analysis were performed in order to investigate the onset of convection for fixed Prandtl number Pr = 1,100; relaxation time F = 0.1,1,100 and retardation time E = 0.05,0.1.

One main conclusion of this work is that proportional control was not successful for all viscoelastic fluids and this was due to the time dependent fluid motions. Since convection sets in as oscillatory motions the controller gain K may join the viscoelasticity to reduce the oscillations (Pr = 100,F = 0.1,E = 0) but under certain conditions K may drive oscillatory motions too (Pr = 1,F = 10,E = 0). This finding indicates that a further correction may be needed, to delay the onset of convection, in the form of a proportional integral controller. On the other hand, it was found that control gains larger than K = 1 can make the dynamics of the system more complex than it is.

Another important conclusion is related to the viscoelastic response. Relaxation and retardation time are dramatically influenced by the controller gain since for viscoelastic Jeffreys fluids both, E and F, may be opposed to the onset of convection a in the case of Pr = 10 where the stabilizing effect of K increases from F = 1 to F = 10. However, for Pr = 10 and E = 0 the control gain K has a negative effect on the hydrodynamics of Maxwell fluids with F = 1 while for the case of F = 10 the critical Rayleigh number increases.

Perhaps, an interesting matter of future investigations could be the effect of proportional control on the convective pattern selection in viscoelastic fluids confined between poorly conducting boundaries which are a convenient case for analytical investigations.

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